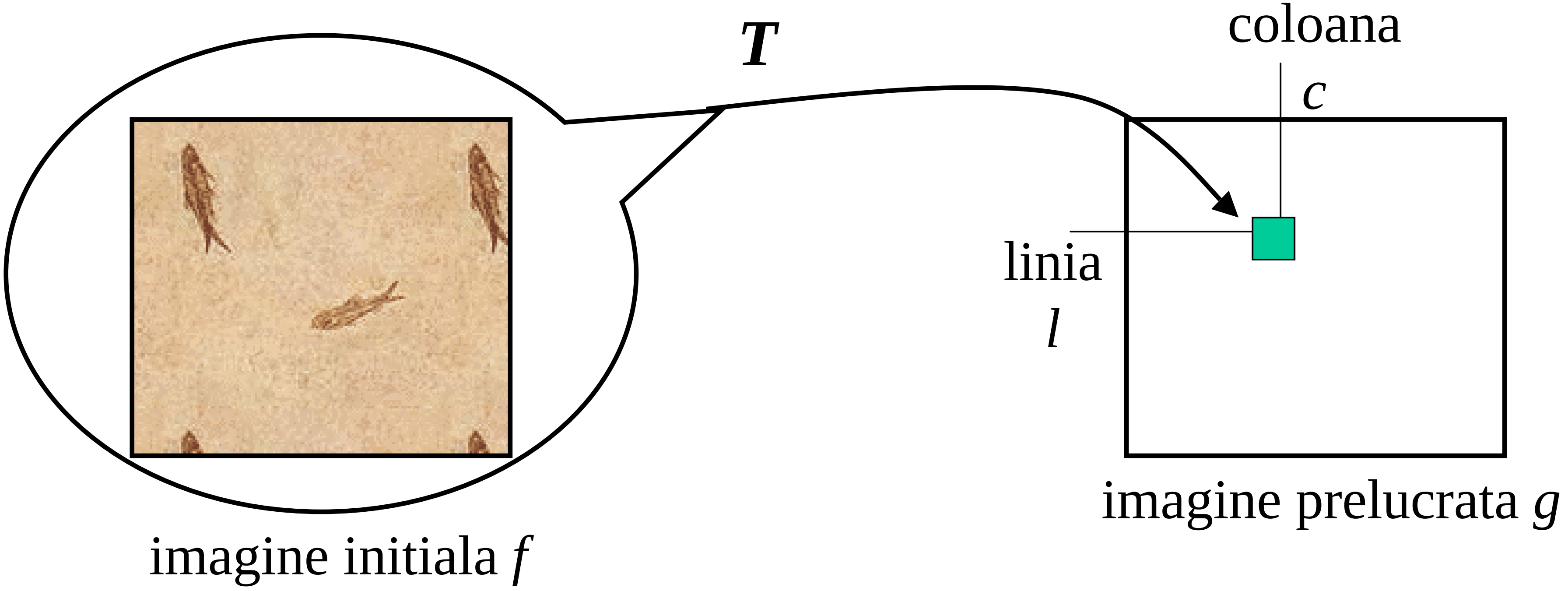
TRANSFORMARI INTEGRALE UNITARE

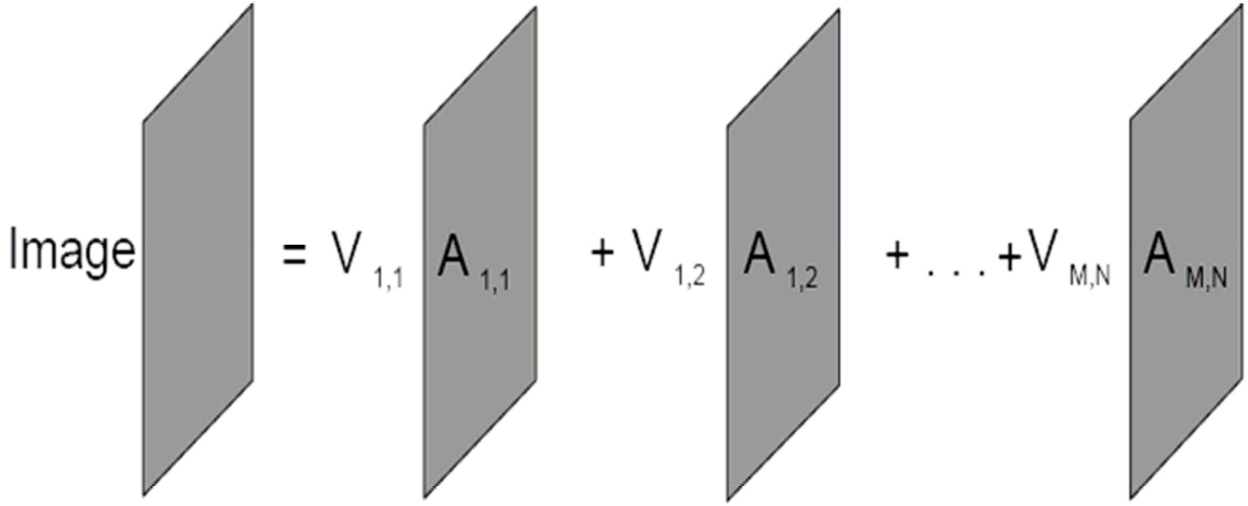
C. VERTAN

Operatii integrale



Noua valoare a oricarui pixel din imaginea prelucrata rezulta din combinarea valorilor tuturor ale pixelilor din imaginea initiala.

Transform techniques



Representation of an image by orthogonal image functions

$$\underset{\text{Image}}{\begin{bmatrix} 6 & 0 \\ 4 & 6 \end{bmatrix}} = \underset{V_{11}}{4}\underset{\mathbf{A}_{11}}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} + \underset{V_{12}}{1}\underset{\mathbf{A}_{12}}{\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}} - \underset{V_{21}}{1}\underset{\mathbf{A}_{21}}{\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}} + \underset{V_{22}}{2}\underset{\mathbf{A}_{22}}{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

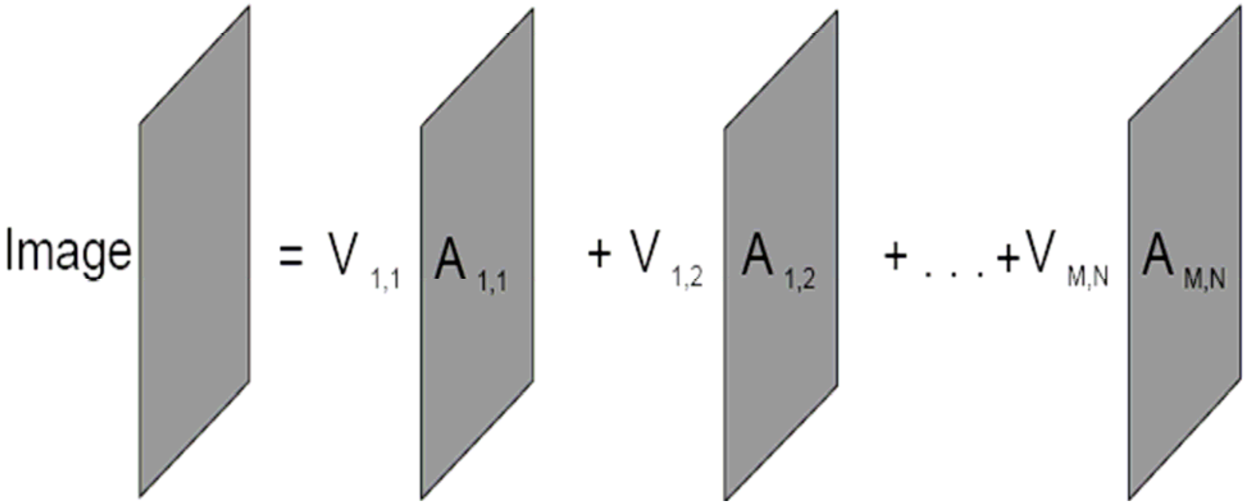
Basic images

The image U can be represented by a series representation

$$U = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} v(k,l) A_{k,l}$$

$A_{k,l}$ Basic image

$$v(k,l) = \langle U, A_{k,l} \rangle$$



Bi-dimensional transforms

$$v(k,l)=\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}u(m,n)a_{k,l}(m,n);\quad 0\le \mathrm{k,l}\le \mathrm{N-1}$$

Direct transform

$$u(m,n)=\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}v(k,l)a_{k,l}(m,n);\quad 0\le \mathrm{m,n}\le \mathrm{N-1}$$

Inverse transform

$$\left\{a_{k,l}(m,n)\right\}$$

Image transform

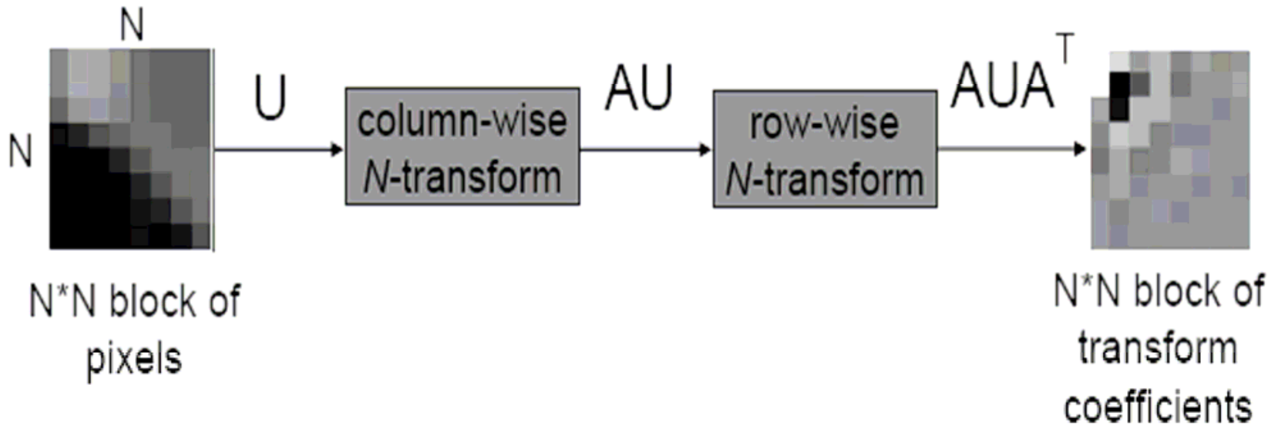
Basic discrete orthonormal functions $\rightarrow$ $$\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}a_{k,l}(m,n)a_{k',l'}(m,n)=\delta(k-k',l-l')$$

Complete discrete orthonormal functions $\rightarrow$ $$\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}a_{k,l}(m,n)a_{k,l}(m',n')=\delta(m-m',n-n')$$

Separable unitary transforms

$$V = AUA^T$$
$$U = A^T VA$$

$\Rightarrow$ Easy implementation



Unidimensional transforms

$$Au = v \quad v(k) = \sum_{n=0}^{N-1} a(k,n)u(n); \;\; 0 \le \mathrm{k} \le \mathrm{N-1}$$

Direct transform

$$A^{-1} = A^T$$

A unitary matrix

$$A^T v = u \quad u(n) = \sum_{k=0}^{N-1} v(k)a(k,n); \;\; 0 \le \mathrm{n} \le \mathrm{N-1}$$

Inverse transform

$$a_k = \{a(k,n),\; 0 \le \mathrm{n} \le \mathrm{N-1}\}^T$$

Base vectors of A = columns of A^T

Unitary transforms properties

1. Conservation of the energy and rotation

Parseval's Theorem $$\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\left|u(m,n)\right|^2 = \sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\left|v(k,l)\right|^2$$

2. Compaction of energy and variances

$$\mu_v = E\{v\} = E\{Au\} = AE\{u\} = A\mu_u$$
$$R_v = E\{(v-\mu_v)(v-\mu_v)^T\} = AR_uA^T$$
$$\sigma_v^2(k) = \{R_v\}_{k,k} = \{AR_uA^T\}_{k,k}$$

3. Decorrelation

Decorelarea si concentrarea energiei

$$E_{\mathbf{v}} = \|\mathbf{v}\|^2 = \mathbf{v}^{*T}\mathbf{v} = (\mathbf{A}\mathbf{u})^{*T}\mathbf{A}\mathbf{u} = \mathbf{u}^{*T}\mathbf{A}^{*T}\mathbf{A}\mathbf{u} = \mathbf{u}^{*T}\mathbf{u} = \|\mathbf{u}\|^2 = E_{\mathbf{u}}$$

$$\mathrm{C}_{\mathbf{v}} = \overline{(\mathbf{v}-\overline{\mathbf{v}})(\mathbf{v}-\overline{\mathbf{v}})^{*T}} = \overline{\mathbf{A}(\mathbf{u}-\overline{\mathbf{u}})(\mathbf{u}-\overline{\mathbf{u}})^{*T}\mathbf{A}^{*T}} = \mathbf{A}\mathbf{C}_{\mathbf{u}}\mathbf{A}^{*T}$$

Ce imi arata matricea de autocovariatie $\mathbf{C}_{\mathrm{u}}$?

- variantele componentelor pe diagonala principala
- corelatia dintre componente pe diagonalele secundare

Decorelare: matricea de autocovariatie sa fie diagonala.

Transformari unitare discrete separabile

(implicit sunt fixe – sunt definite de aceiasi coeficienti pentru transformarea oricarui semnal)

transformata Fourier
transfomata Cosinus
transformata Sinus

Transformari unitare discrete adaptive

(coeficienti depedenti de valorile semnalului prelucrat)

transformata Karhunen-Loeve

Transformarea Fourier discreta (bidimensionala)

C. VERTAN

- Each image can be decomposed in weighted sum of complex exponentials (sines and cosines) of frequency *f* and angle ϕ. (or two frequency components *u* and *v*)

$$g(x,y)=\frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}G(u,v)e^{j\frac{2\pi}{N}(ux+vy)}$$

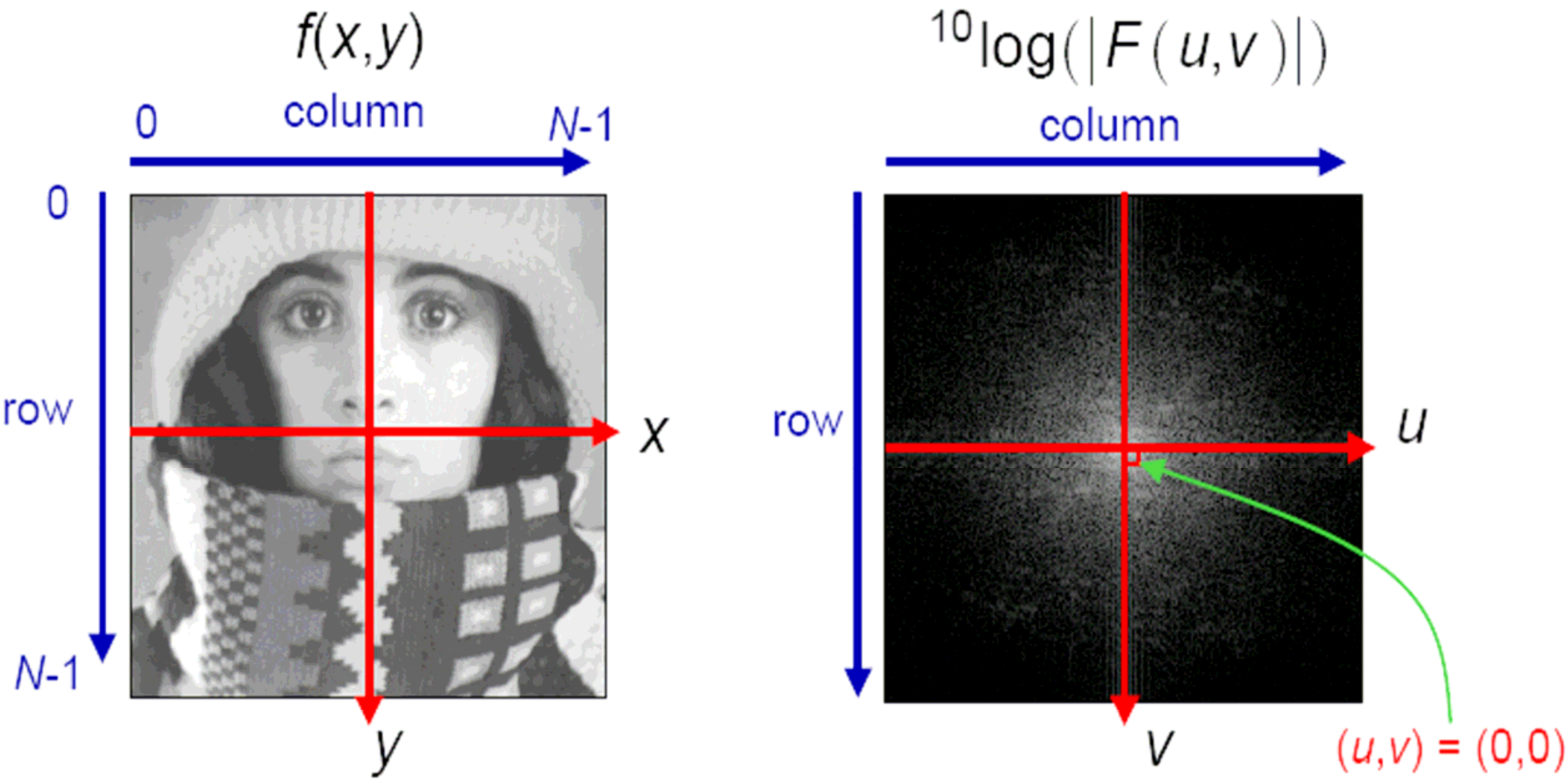
image size
NxN

$$G(u,v)=\frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}g(x,y)e^{-j\frac{2\pi}{N}(ux+vy)}$$

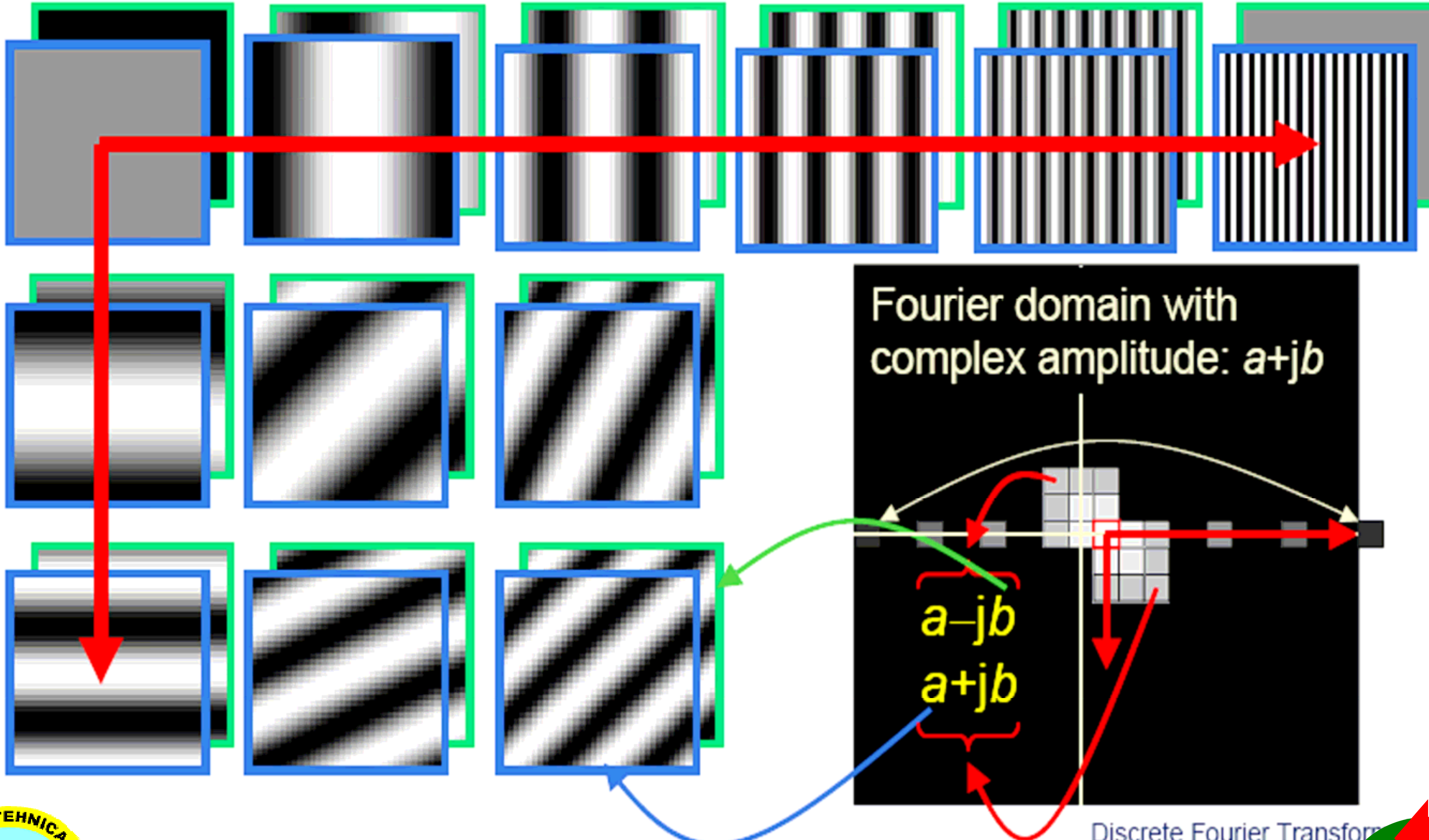
- For real-valued images: $Ev\{g(x,y)\}\xleftrightarrow{F}Re\{G(u,v)\}$

$$Od\{g(x,y)\}\xleftrightarrow{F}j\,Im\{G(u,v)\}$$

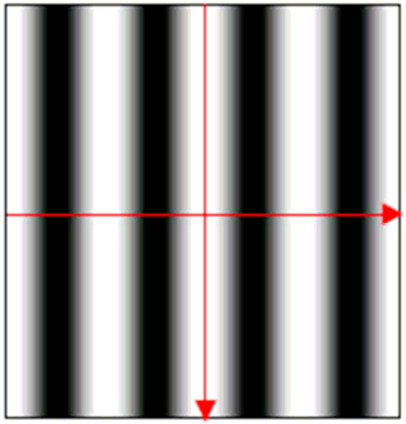
- $F(u,v)$ is the complex amplitude of the eigenfunction $\exp(j\,(2\pi/N)(ux+vy))$
  Note that $\mathbf{exp}(j\,(2\pi/N)(ux+vy)) = \mathbf{cos}((2\pi/N)(ux+vy)) + j\,\mathbf{sin}((2\pi/N)(ux+vy))$
- Standard display is the logarithm of the magnitude: $\log(|F(u,v|)$



An image is a weighted sum of cos (even) and sin (odd) images.



Functii proprii: pare si impare

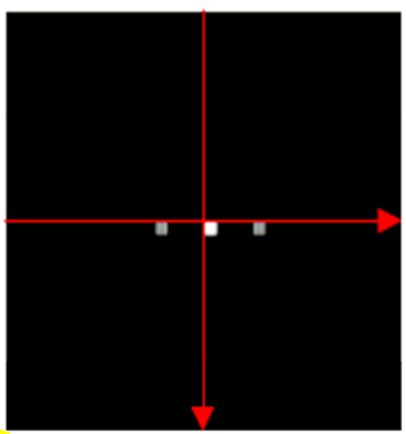


Real & even

$$1+\cos\left(\frac{2\pi}{N}u_0x\right)=$$

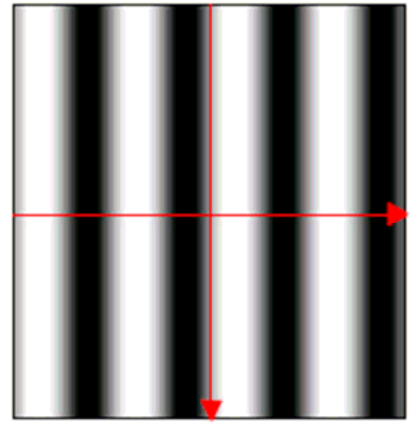
$$1+\frac{e^{j\frac{2\pi}{N}u_0x}+e^{j\frac{2\pi}{N}(-u_0)x}}{2}$$

$\Updownarrow \mathcal{F}$



Real & even

$$\frac{1}{N}\left(\begin{array}{l}\delta(u,v)+\\ \frac{1}{2}\delta(u-u_0,v)+\\ \frac{1}{2}\delta(u+u_0,v)\end{array}\right)$$

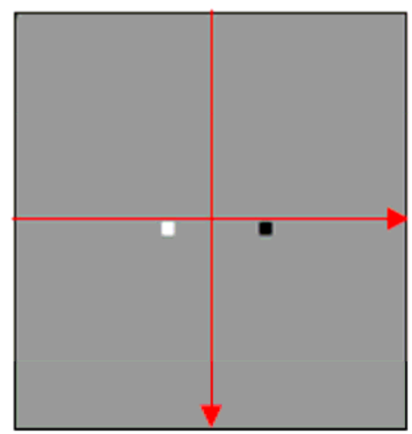


Real & odd

$$\sin\left(\frac{2\pi}{N}u_0x\right)=$$

$$\frac{e^{j\frac{2\pi}{N}u_0x}-e^{j\frac{2\pi}{N}(-u_0)x}}{2j}$$

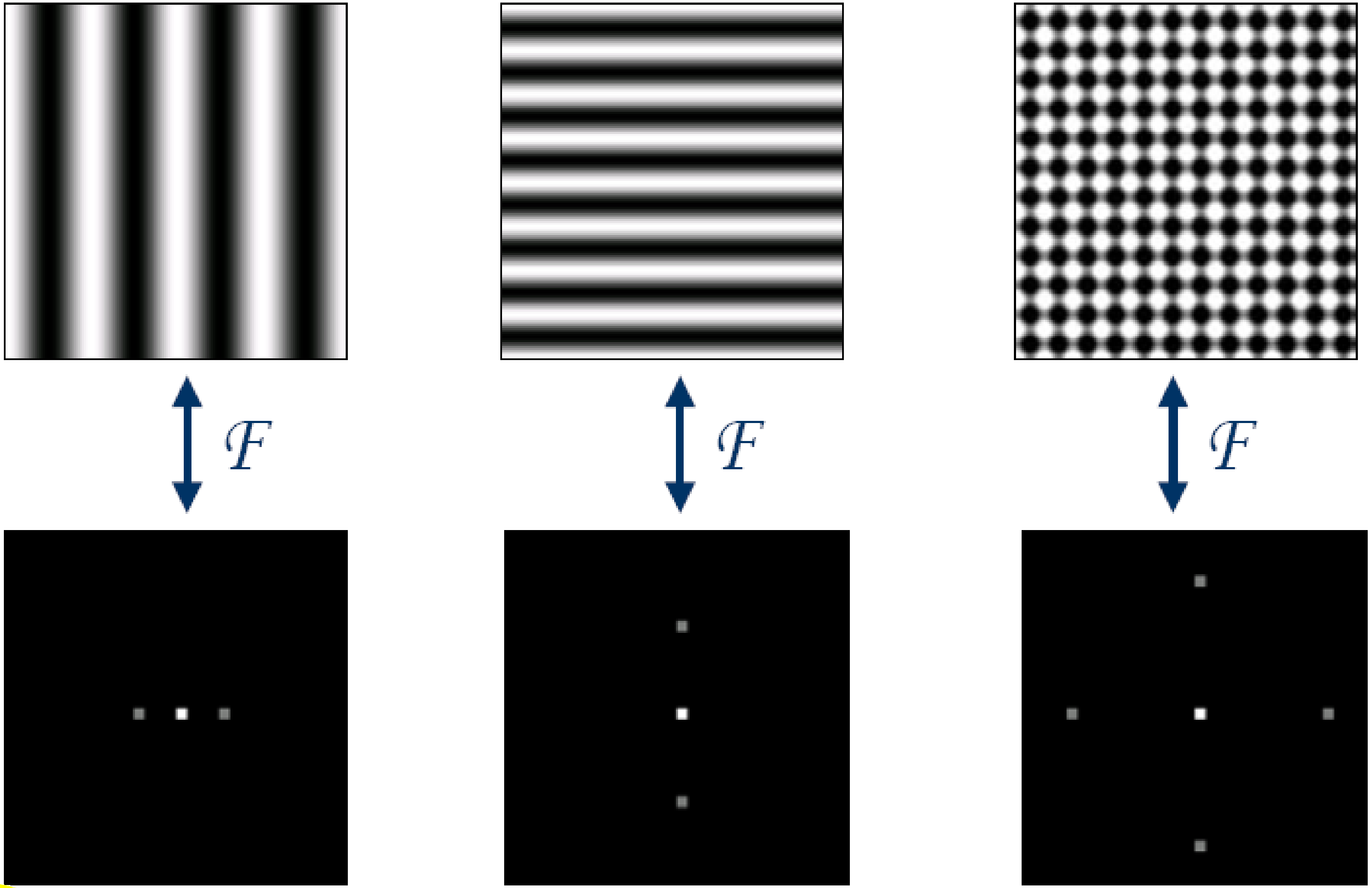
$\Updownarrow \mathcal{F}$



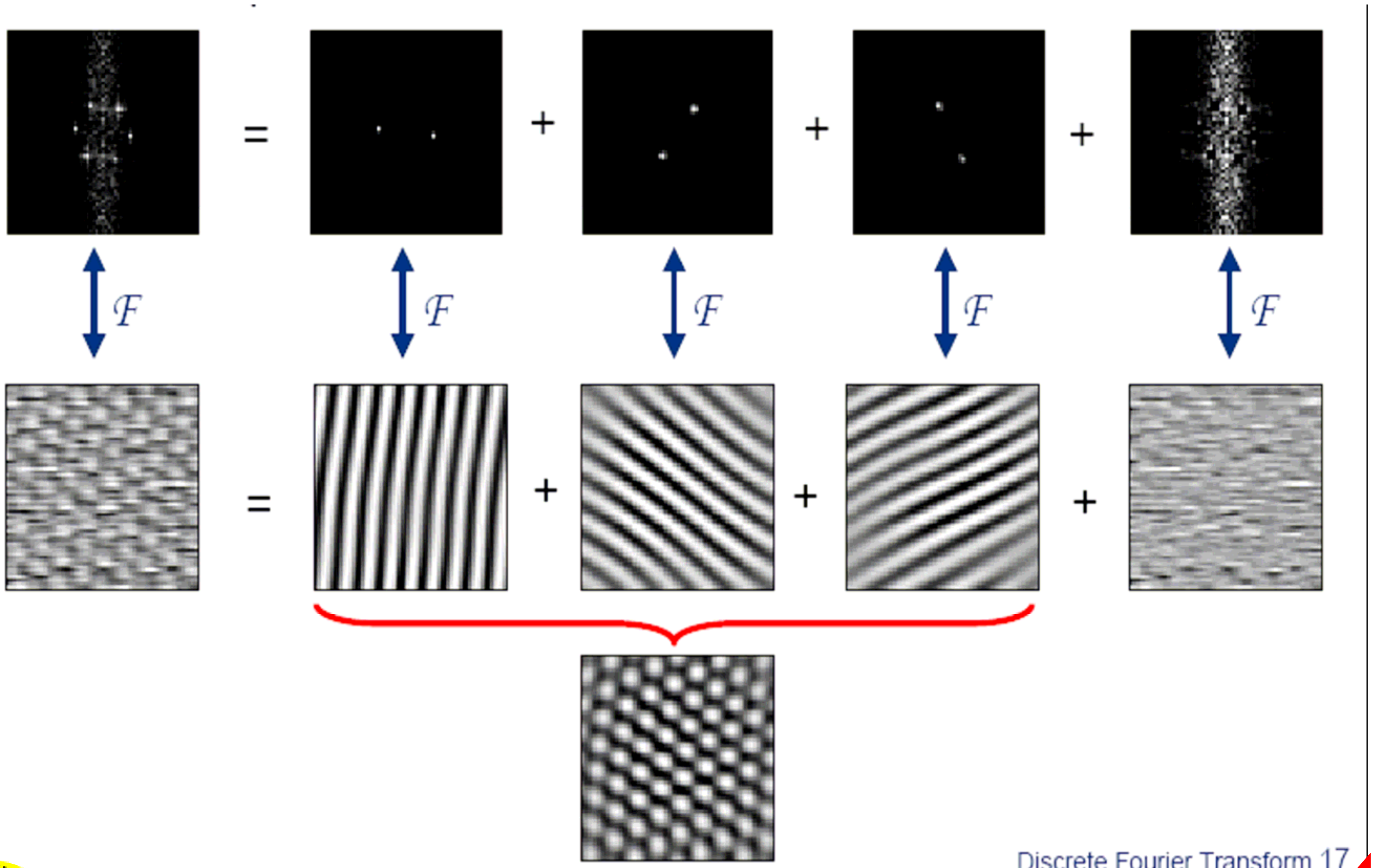
Imag & odd

$$\frac{1}{N}j\left(\begin{array}{l}-\frac{1}{2}\delta(u-u_0,v)\\ +\frac{1}{2}\delta(u+u_0,v)\end{array}\right)$$

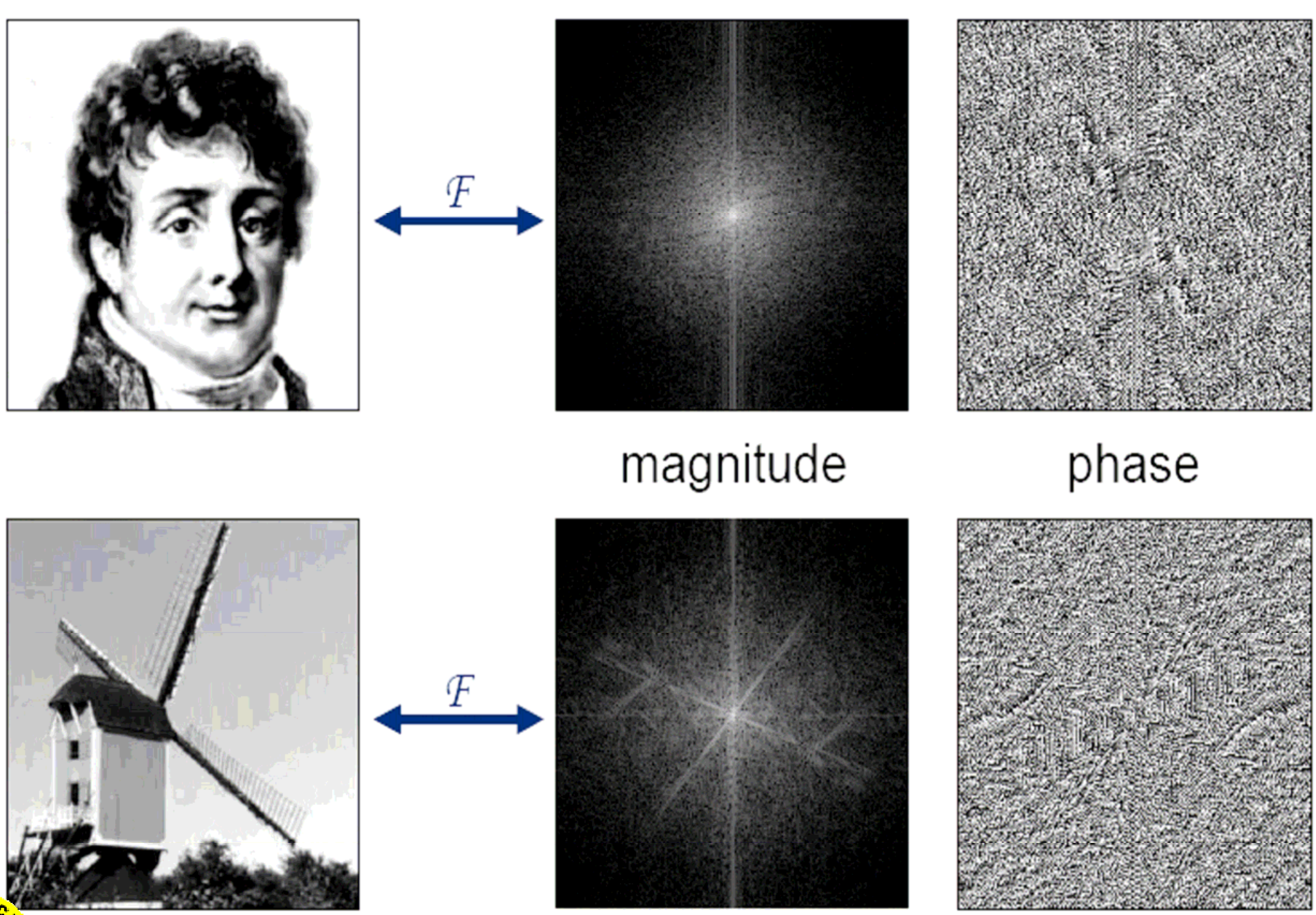
Orientarea si frecventa



Superpozitia



C. VERTAN

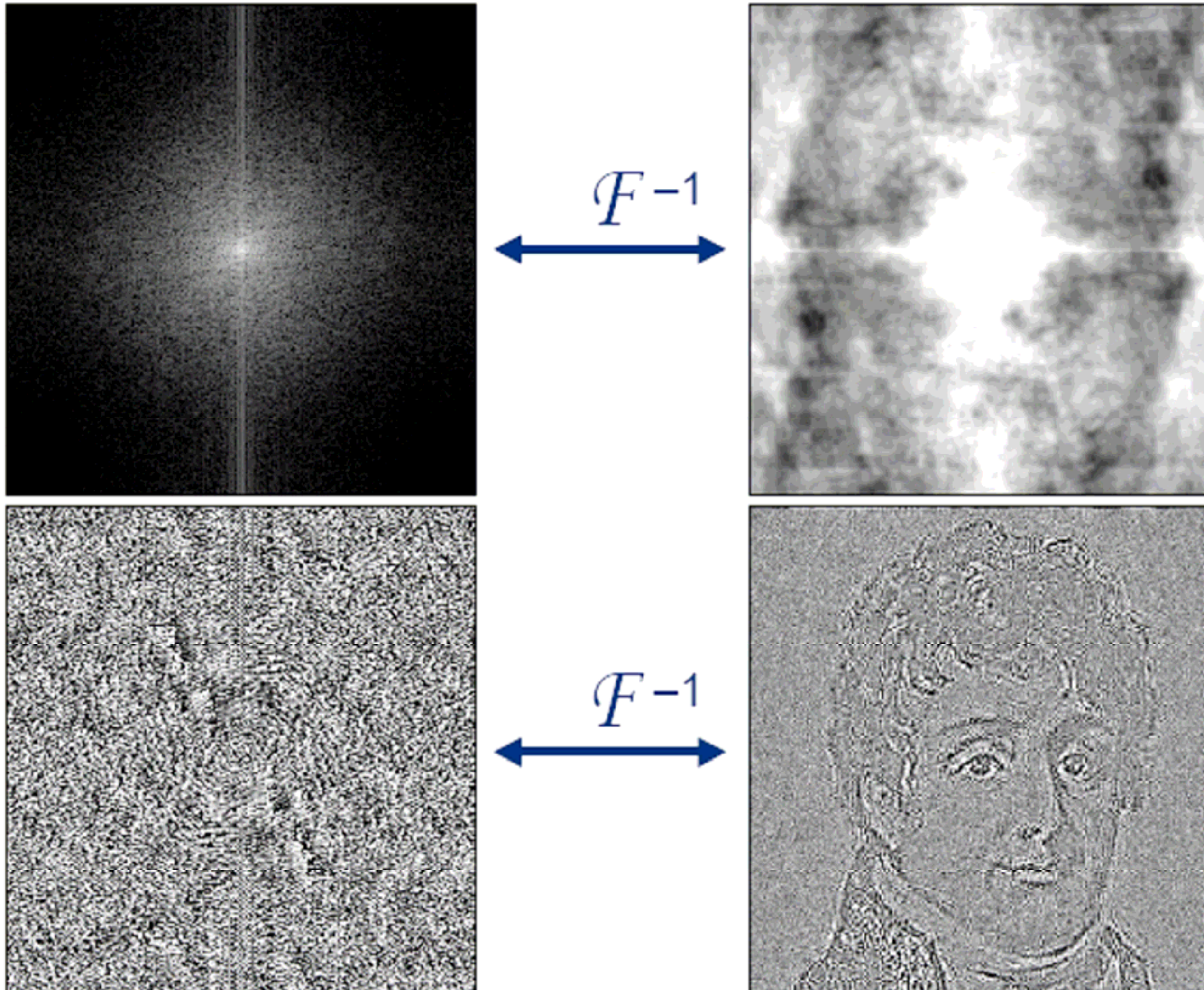


$\mathcal{F}$
$\mathcal{F}$
magnitude
phase

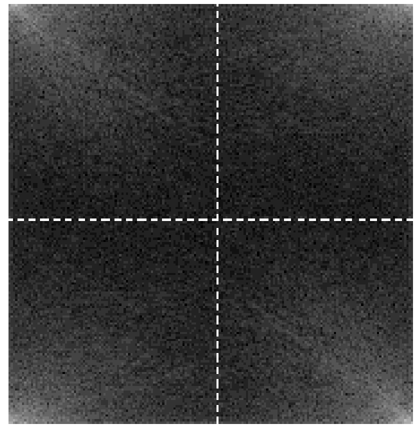
Modulul si faza

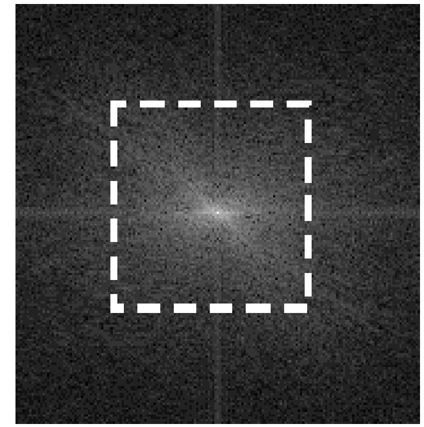
$$|F(u,v)|$$

$$e^{j\sphericalangle F(u,v)} = \frac{F(u,v)}{|F(u,v)|}$$



Compactarea energiei





Size 256x256

DFT Transform

This subimage (128x128) contributes to 68.7% to the total energy of the image.

(Poor compaction capacity)

Inverse Transform

Proprietati fundamentale

convolutia $$f \otimes g = T^{-1}(F * G)$$

convolutia in domeniul spatial este transformat prin Fourier in produs (punct cu punct) in domeniul de frecventa.

transformata rapida

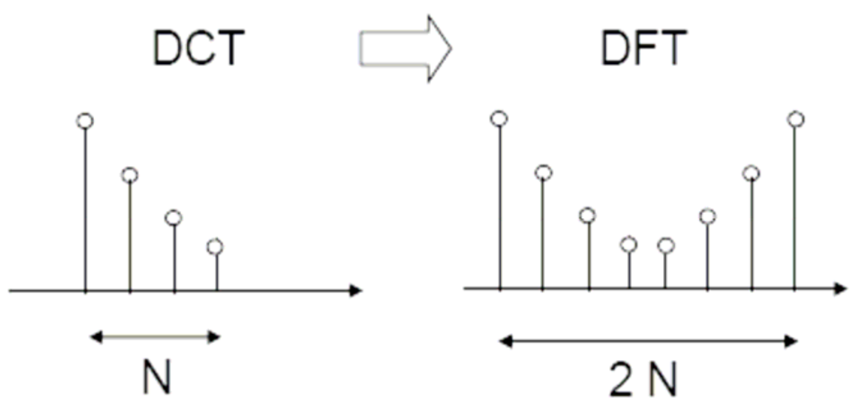
FFT – fast Fourier transform

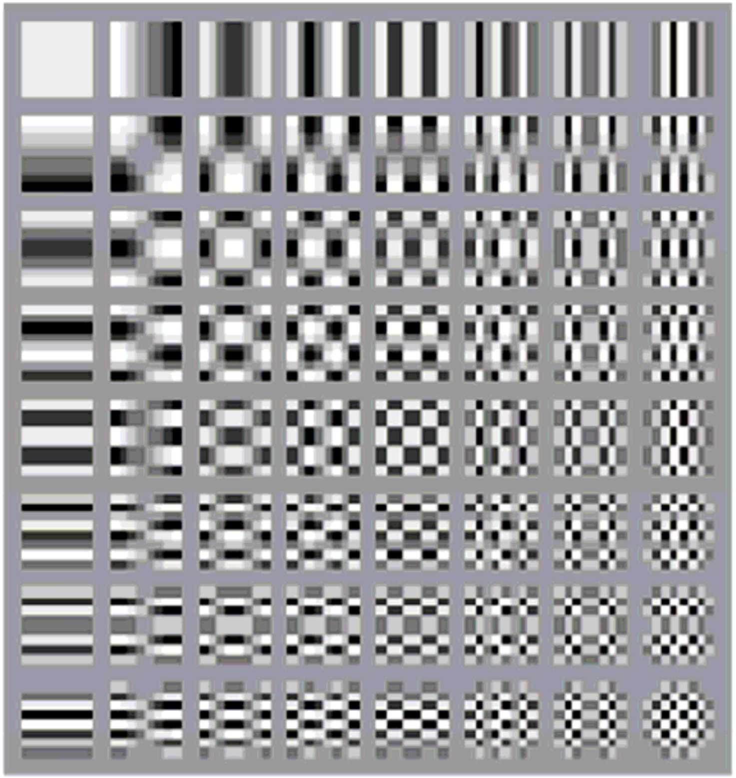
Discrete CosineTransform - DCT

$$F(u,v)=\frac{1}{4}C(u)C(v)\left[\sum_{x=0}^{7}\sum_{y=0}^{7}f(x,y)\cos\frac{(2x+1)u\pi}{16}\cos\frac{(2y+1)v\pi}{16}\right]$$

$$C(u),C(v)=\frac{1}{\sqrt{2}}\quad \text{for } u,v=0$$

$$C(u),C(v)=1\quad \text{for the other cases}$$





Basic DCT Images

Transformarea Sinus Discreta

Transformările directă şi inversă a unei secevenţe **u** sunt definite ca:

$$v(k) = \sqrt{\frac{2}{N+1}} \sum_{n=0}^{N-1} u(n) \sin \frac{\pi(k+1)(n+1)}{N+1}, \; k = \overline{0, N-1}$$

$$u(n) = \sqrt{\frac{2}{N+1}} \sum_{k=0}^{N-1} v(k) \sin \frac{\pi(k+1)(n+1)}{N+1}, \; n = \overline{0, N-1}$$

Elementele matricii **S** sunt date de

$$S(k,n) = \sqrt{\frac{2}{N+1}} \sin \frac{\pi(k+1)(n+1)}{N+1}, \; k,n = \overline{0, N-1}$$

Decorelarea si concentrarea energiei

$$E_{\mathbf{v}} = \|\mathbf{v}\|^2 = \mathbf{v}^{*T}\mathbf{v} = (\mathbf{A}\mathbf{u})^{*T}\mathbf{A}\mathbf{u} = \mathbf{u}^{*T}\mathbf{A}^{*T}\mathbf{A}\mathbf{u} = \mathbf{u}^{*T}\mathbf{u} = \|\mathbf{u}\|^2 = E_{\mathbf{u}}$$

$$\mathbf{C}_{\mathbf{v}} = \overline{(\mathbf{v} - \overline{\mathbf{v}})(\mathbf{v} - \overline{\mathbf{v}})^{*T}} = \overline{\mathbf{A}(\mathbf{u} - \overline{\mathbf{u}})(\mathbf{u} - \overline{\mathbf{u}})^{*T}\mathbf{A}^{*T}} = \mathbf{A}\mathbf{C}_{\mathbf{u}}\mathbf{A}^{*T}$$

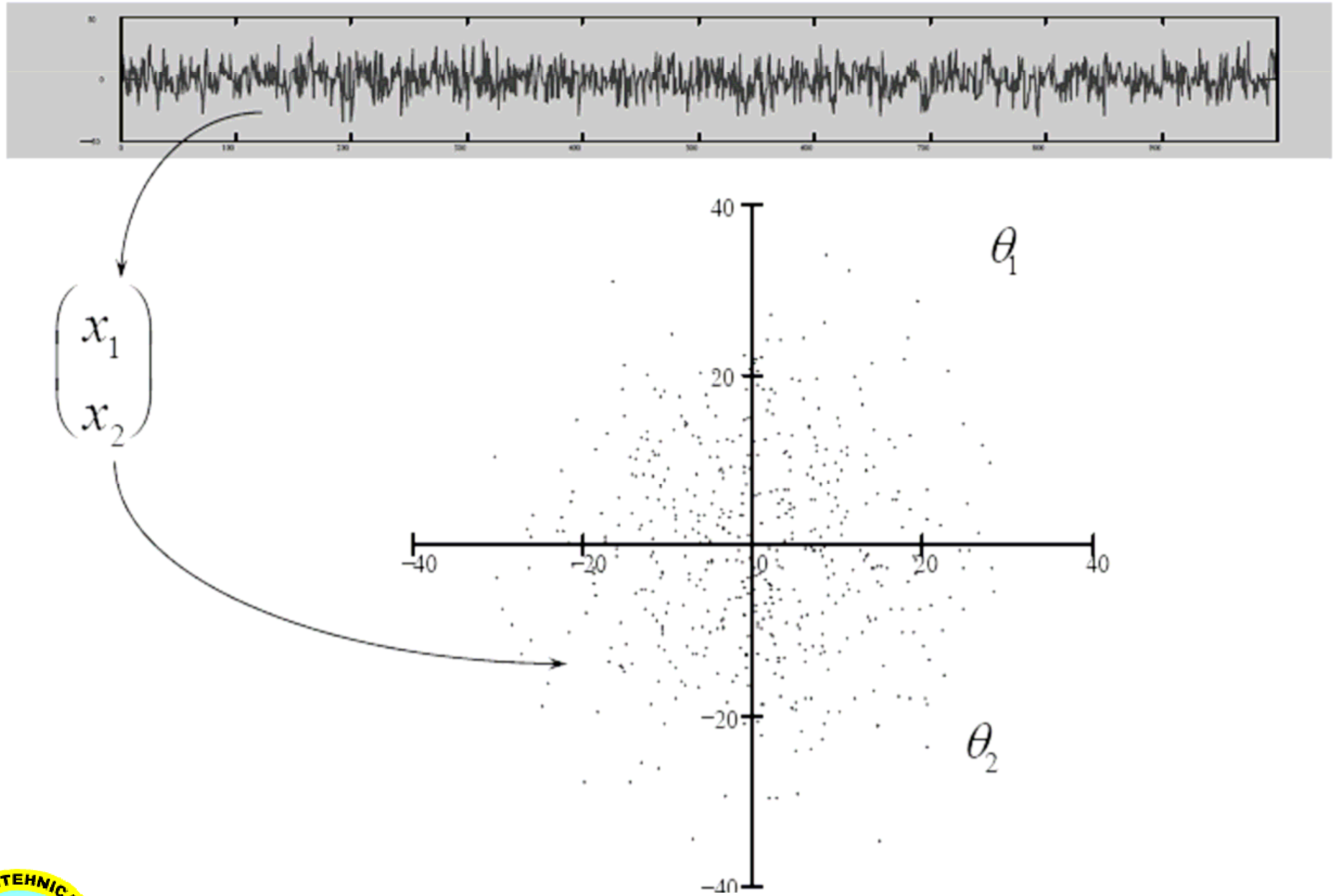
Ce imi arata matricea de autocovariatie $\mathbf{C}_u$?

- variantele componentelor pe diagonala principala
- corelatia dintre componente pe diagonalele secundare

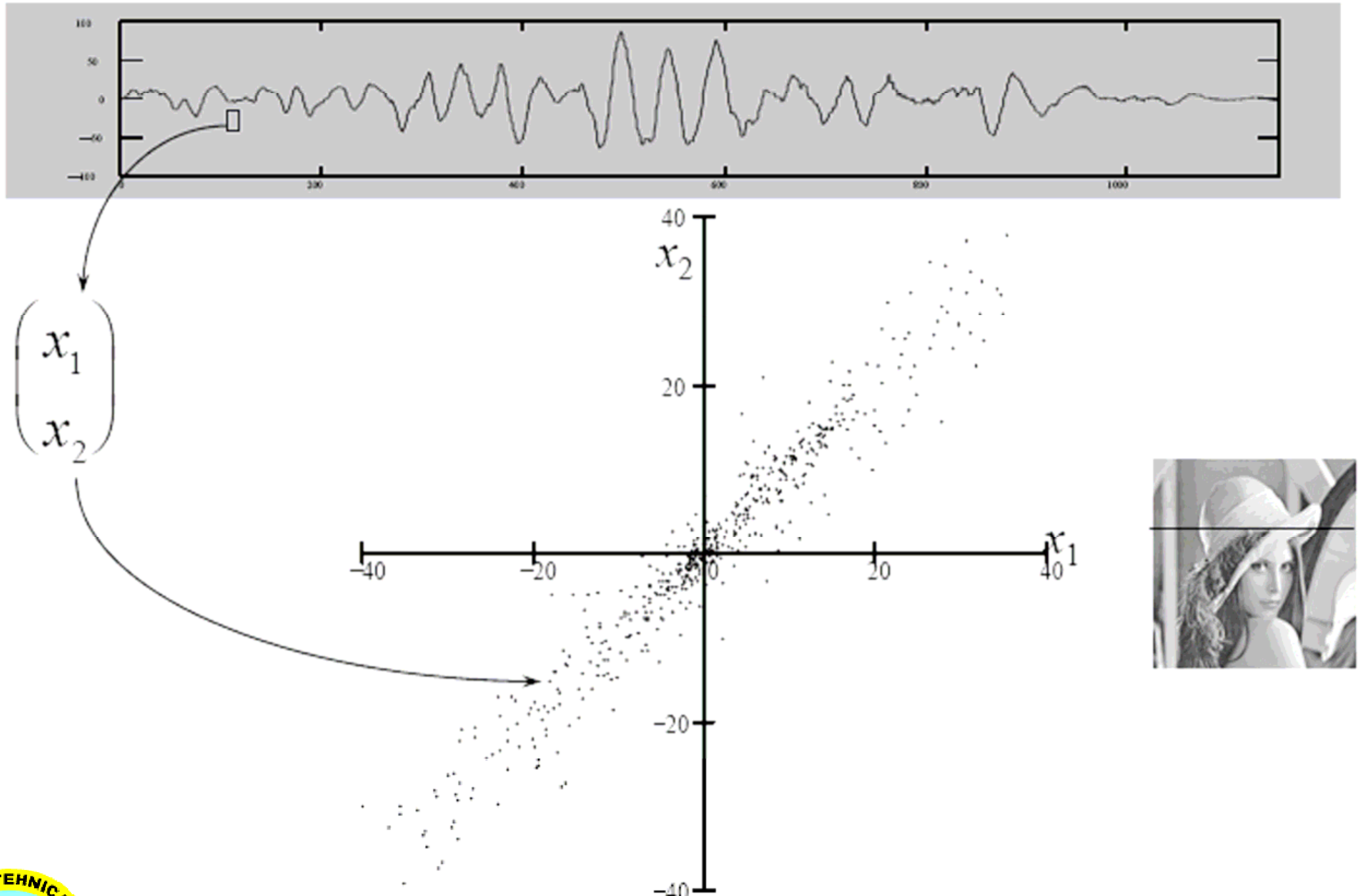
Decorelare: matricea de autocovariatie sa fie diagonala.

Transformarea Karhunen - Loeve

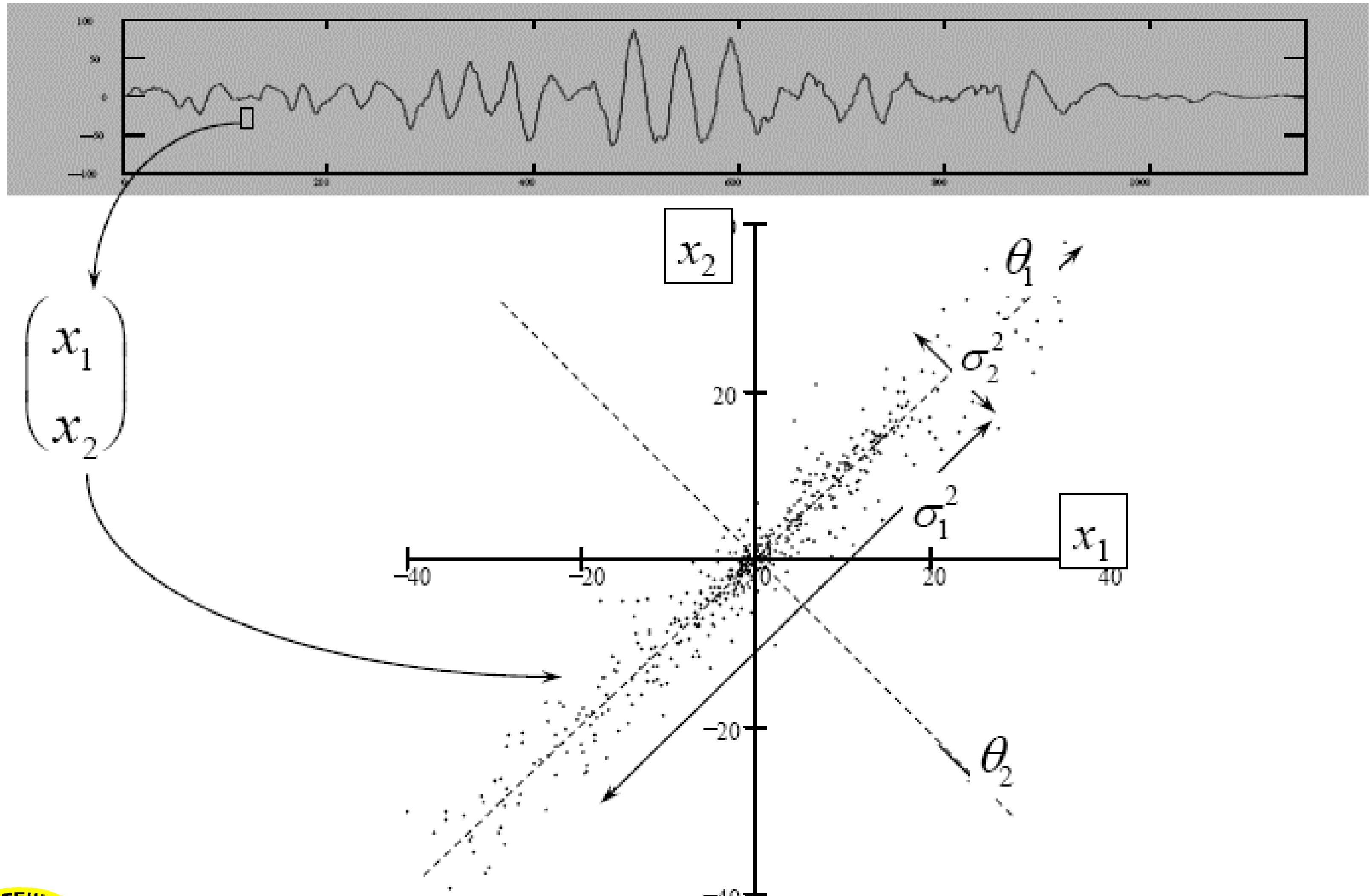
Geometrical interpretation (1)



Geometrical interpretation (2)

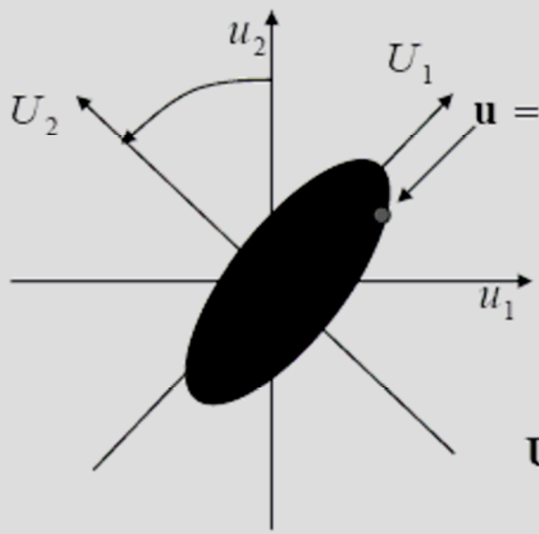


Geometrical interpretation (3)



Geometrical interpretation (4)

- A linear transform can decorrelate random variables
- An orthonormal transform is a rotation of the signal vector around the origin
- Parseval's Theorem holds for orthonormal transforms



$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \mathbf{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

DC basis function

AC basis function

$$\mathbf{U} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \mathbf{Tu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Karhunen - Loeve transform - KLT

$\underline{X}_1 = [x_1,\ldots,x_{64}]$

.
.
.
.

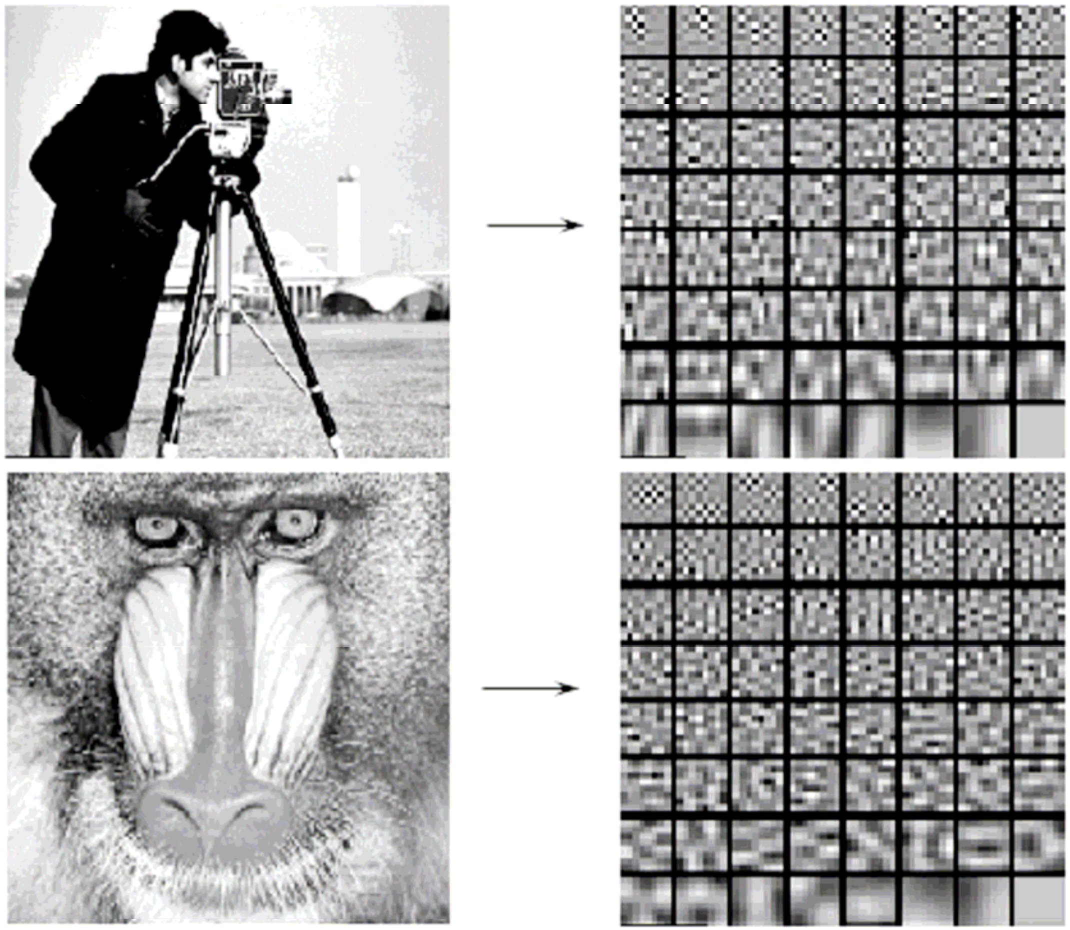
$\underline{X}_N = [x_1,\ldots,x_{64}]$

$$\underline{\underline{R}}_X = E\left\{(\underline{X}-\underline{\mu})(\underline{X}-\underline{\mu})^T\right\}$$

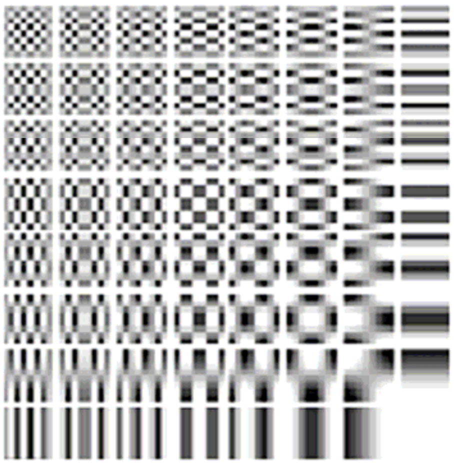
KLT = Eigenvectors of $\underline{\underline{R}}_X$

- Optimum in the Mean Square Error sense (MSE)
- Depends on the input signal

KLT basic images



KLT



DCT

KL e greu de calculat. Ar fi de preferat o aproximare.

Adica: daca avem un semnal, care e transformarea unitara fixa care se apropie cel mai mult de transformata KL a semnalului respectiv ?

Si, inrudit: pentru ce fel de semnale transformarile fixe prezentate sunt transformate optime (KL) ?

Pentru imagini, in general, KL $\cong$ COS;
COS este optimala pentru secvente de valori ce sunt modelate Markov de ordinul 1, cu coeficient de corelatie mare.