ELEMENTE DE MORFOLOGIE MATEMATICA

C. VERTAN

Morfologia matematica

Cadru de abordare diferit:

Pana acum : Imaginea este o functie de doua variabile.

Pixelii imaginii (valori si coordonate de pozitie) sunt structurati in multimi (partitii, forme).

morphos = forma
logos = stiinta

→ stiinta formelor ?

Morfologia matematica

Conduce la caracterizarea formei (multimii ce se prelucreaza) intr-un cadru determinist.

Caracterizarea formei este rezultatul comparatiei (interactiunii, aplicarii de relatii) intre forma necunoscuta si **elementul structurant**.

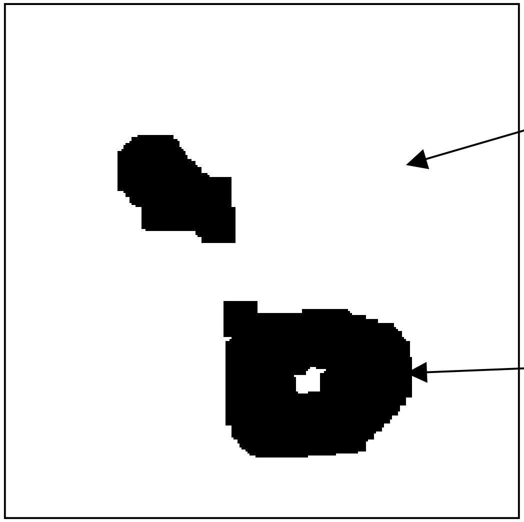
Elementul structurant este o multime geometrica, arbitrara, impusa, cunoscuta. Forma elementului structurant determina proprietatile testate asupra formei necunoscute.

Relatiile aplicate au fost restrinse la operatorii standard ansamblisti (deci la operatiile clasice pe multimi).

Morfologia matematica

Cazul cel mai simplu: imagini binare

Echivalenta imagine – multime este imediata: pixelii a caror valoare este ne-nula formeaza multimea “obiect/ obiecte”; pixelii a caror valoare este nula formeaza multimea “fundal”.



fundal (pixeli de valoare 0)

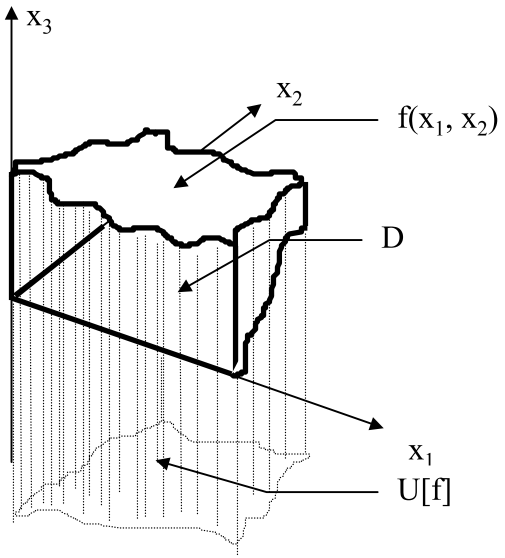
forma (pixeli de valoare 1), multime, obiect ...

Morfologia matematica

Cazul mai complicat: imaginile cu nivele de gri

Imaginile cu nivele de gri sunt reprezentate prin multimi de puncte din $\mathbf{R}^3$; transformarea se numeste <u>umbra</u>.

$$\mathbf{U}(f)=\{(x_1, x_2, x_3)\in\mathbf{R}3 \mid (x_1, x_2)\in \mathrm{D} \text{ si } x_3\leq f(x_1, x_2)\}$$



Morfologia matematica

“Itemurile” de prelucrat: multimi ale caror elemente sunt puncte din $\mathbf{R}^2$ (cazul imaginilor binare) sau $\mathbf{R}^3$ (cazul imaginilor cu nivele de gri).

Un element al unei asemenea multimi (un punct din spatiu) este descris de coordonatele sale:

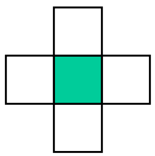
- coordonatele spatiale din suportul plan al imaginii (pentru cazul binar)

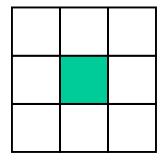
- coordonatele spatiale in suportul imaginii si valoarea nivelului de gri (pentru cazul nivelelor de gri).

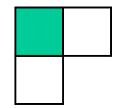
Obiectele (imaginea) de prelucrat si elementul structurant sunt multimi.

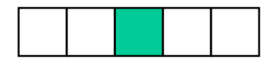
Morfologia matematica

Elementul structurant este echivalentul vecinatatii folosite in operatiile de prelucrare de vecinatate.

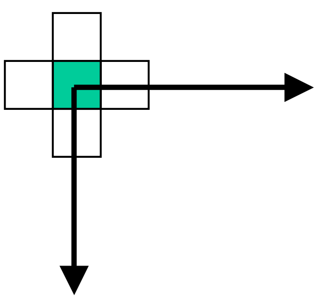
V_4 

V_8 





Elementul structurant are un sistem de coordonate propriu.
(NU SUNT COORDONATELE IMAGINII)



$$V_4 = \{(0,0),(0,1),(0,-1),(1,0),(-1,0)\}$$

Morfologia matematica

Operatorii morfologiei matematice verifica indeplinirea unor relatii intre punctele multimii de prelucrat (obiectul) si elementul structurant.

Relatiile sunt descrise de operatii ansambliste (pe multimi): incluziune, reuniune, intersectie ...

Rezultatul unei operatii morfologice aplicate unei multimi este tot o multime, ale carei puncte specifica pozitiile in care punctele multimii de prelucrat verifica relatia testata de elementul structurant.

Operatiile morfologice de baza

Erodare Θ

Dilatare $\oplus$

$A \Theta B$

$A \oplus B$

multimea A erodata cu elementul structurant B

multimea A dilatata cu elementul structurant B

Intotdeauna elementul structurant ocupa pozitia a doua in operatie.

In general elementul structurant se noteaza cu B.

Erodare

$A \Theta B$

$$A \Theta B = \{\mathbf{x} \mid B_{\mathbf{x}} \subset A\}$$

Erodarea morfologica a multimii *A* prin elementul structurant *B* se defineste ca multimea punctelor (elementelor) cu care (in care) se poate translata elementul structurant astfel incat acesta sa fie inclus in multimea de prelucrat *A* .

Erodare

$A \,\Theta\, B$

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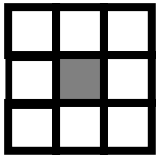
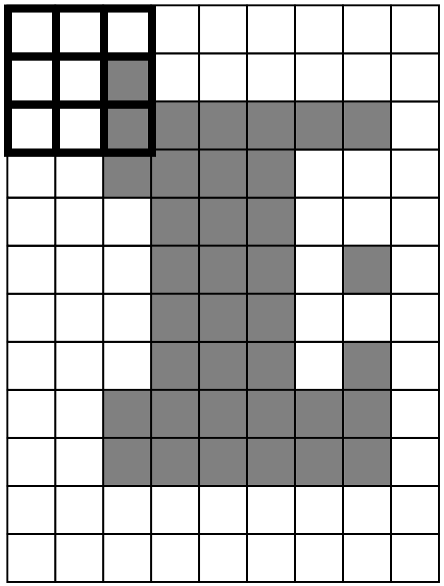
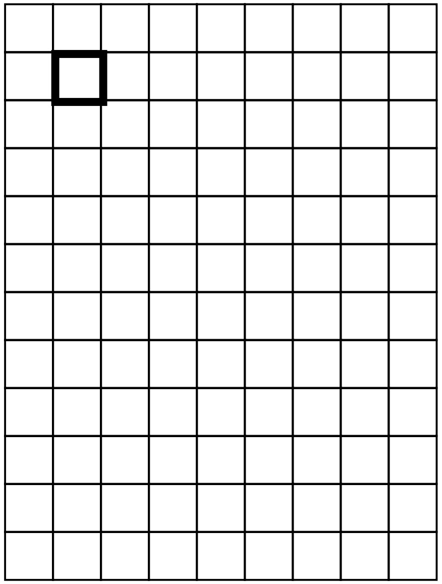
$$B = V_8$$





imagine initiala



imagine prelucrata

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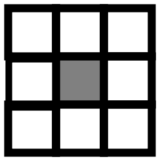

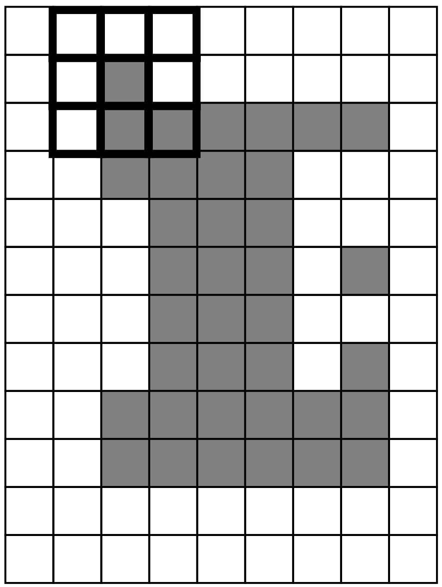

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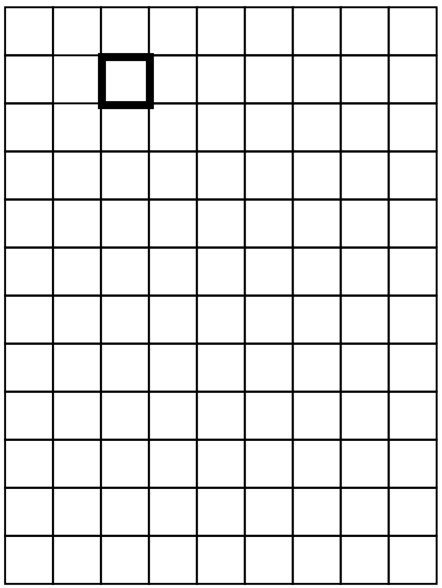

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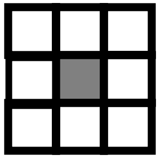

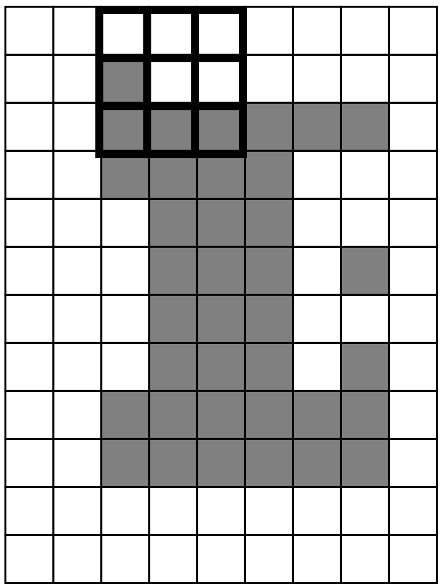

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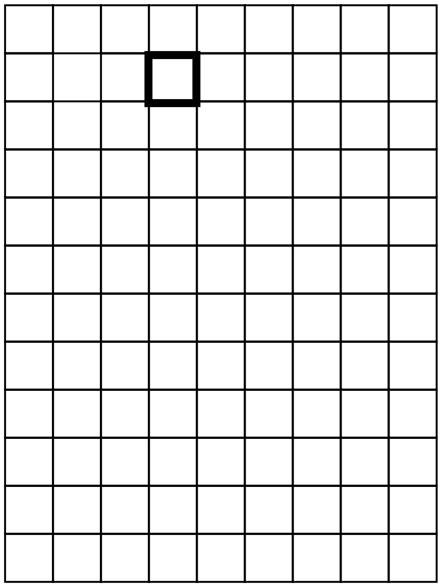

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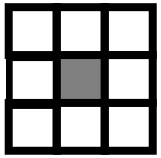

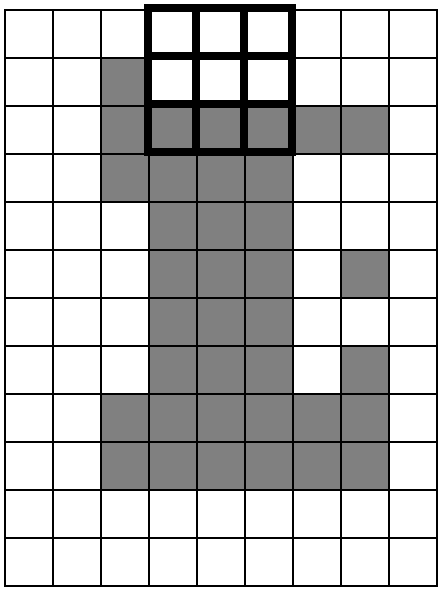

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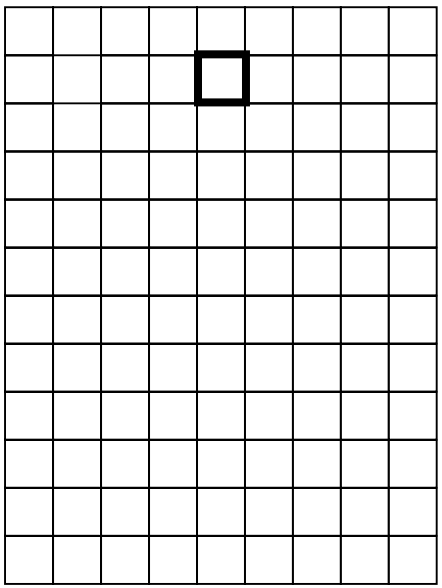

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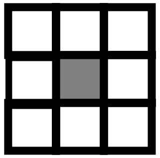

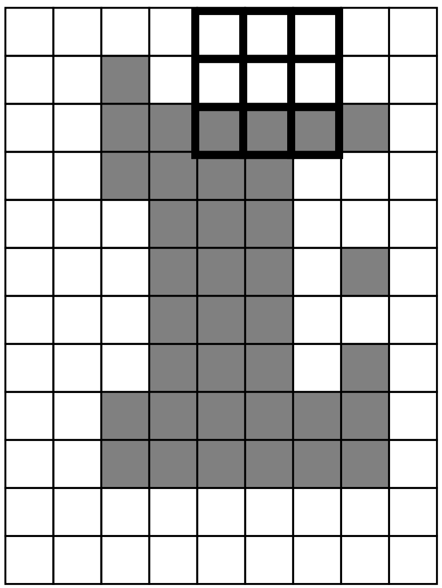

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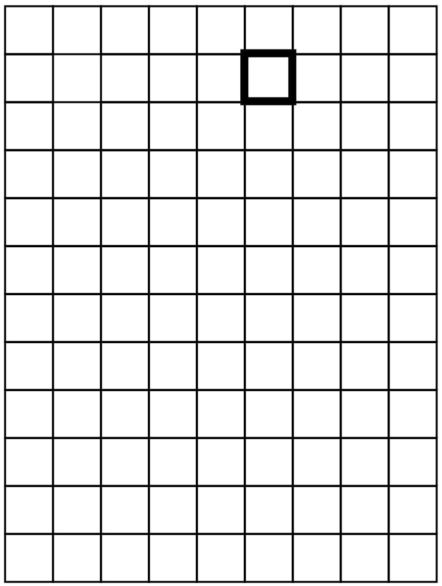

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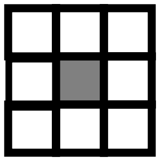

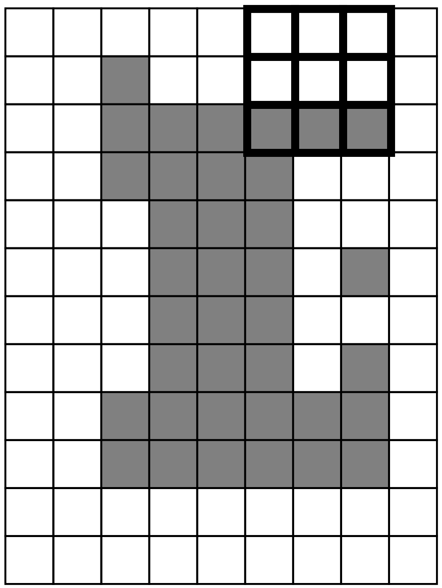

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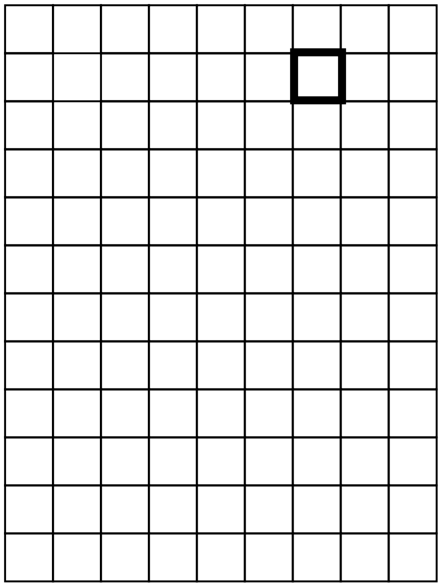

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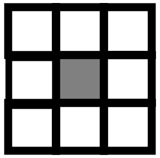

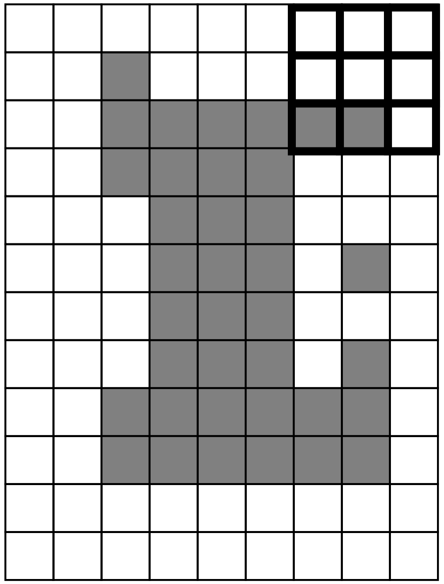

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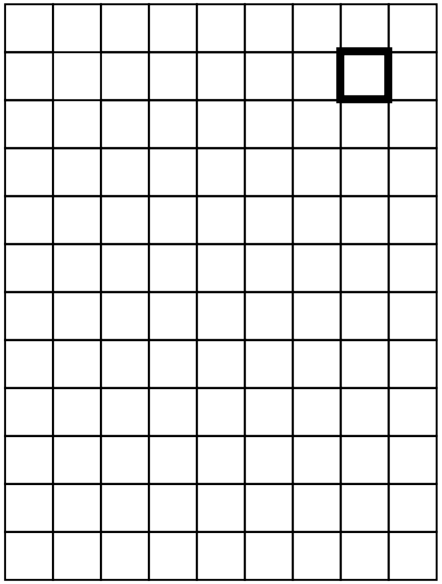

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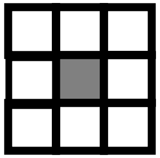

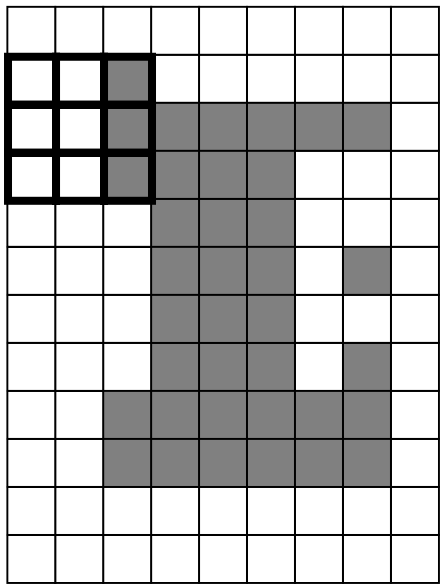

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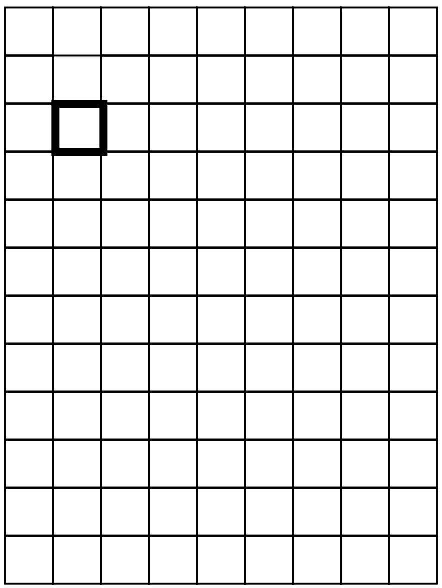

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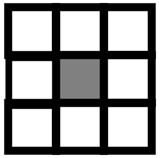

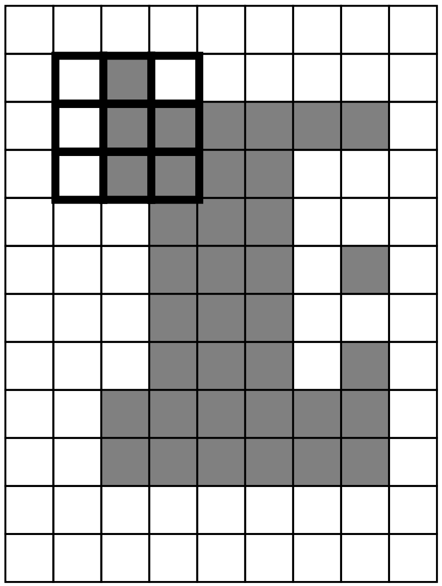

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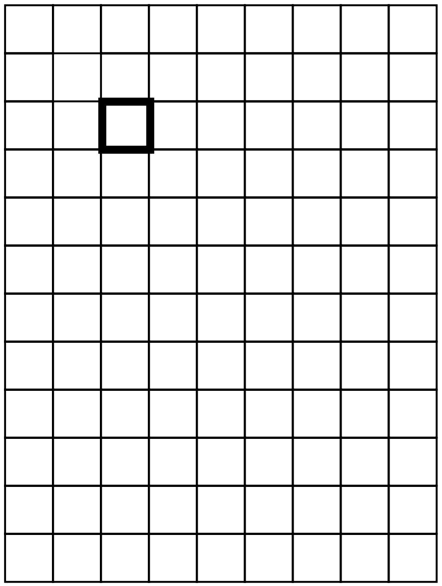

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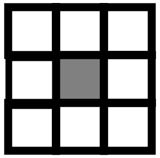

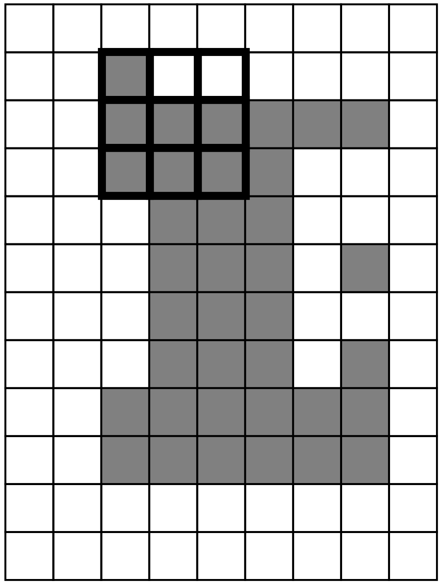

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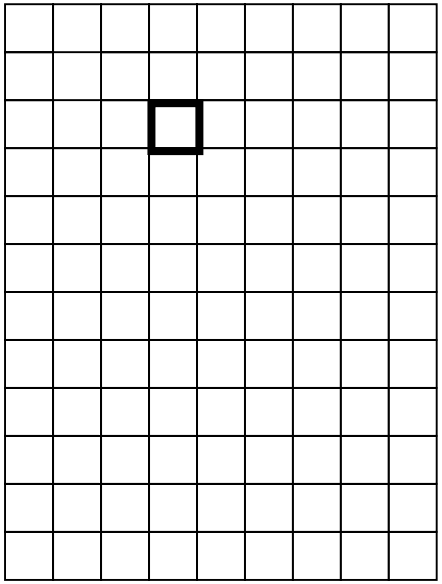

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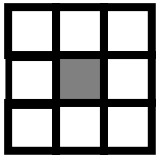
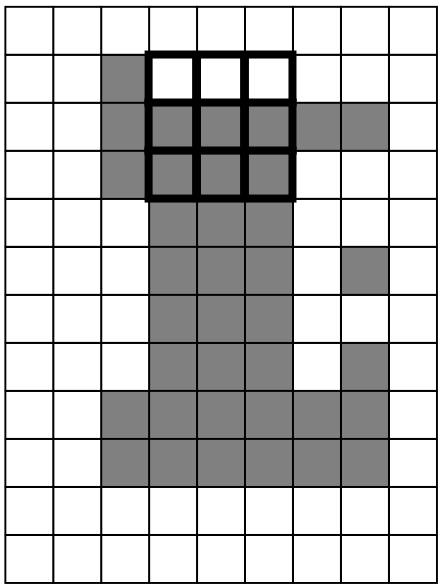
$B = V_8$ 



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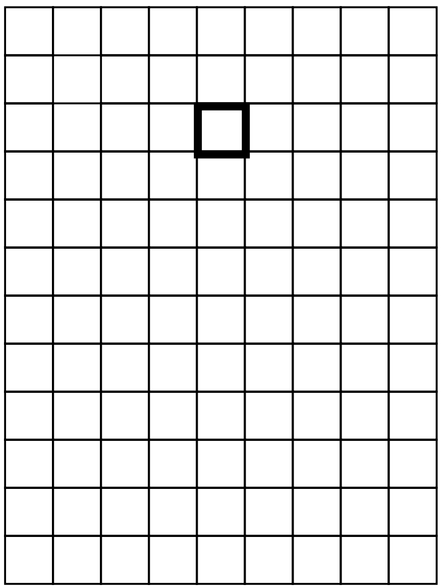

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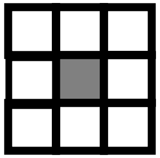

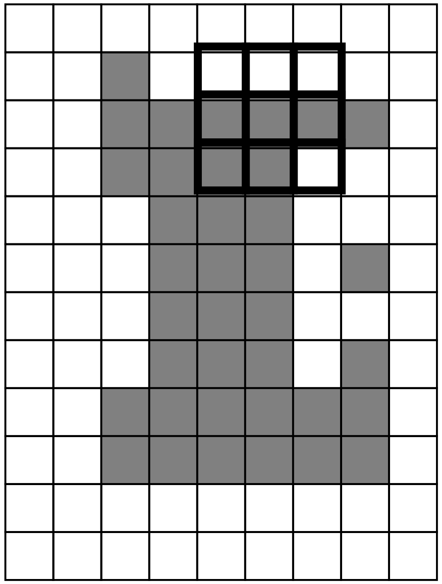

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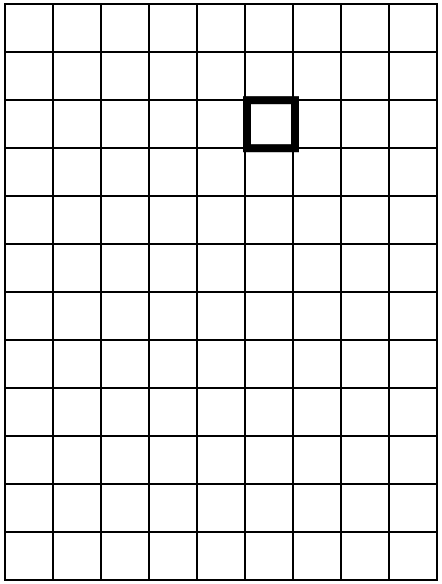

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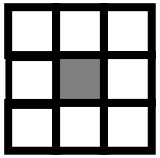

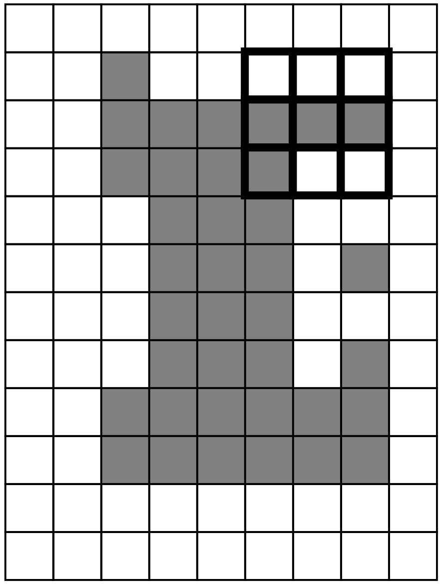

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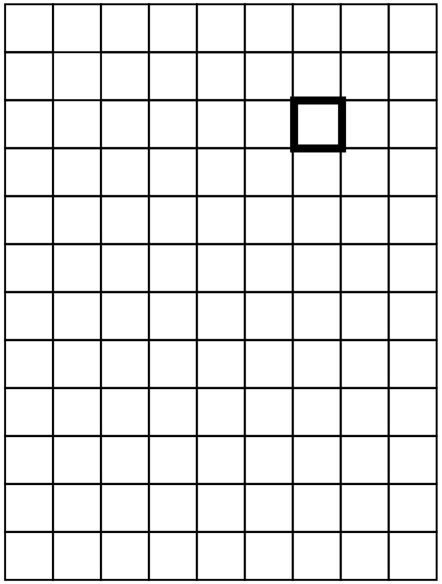

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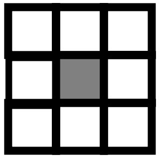

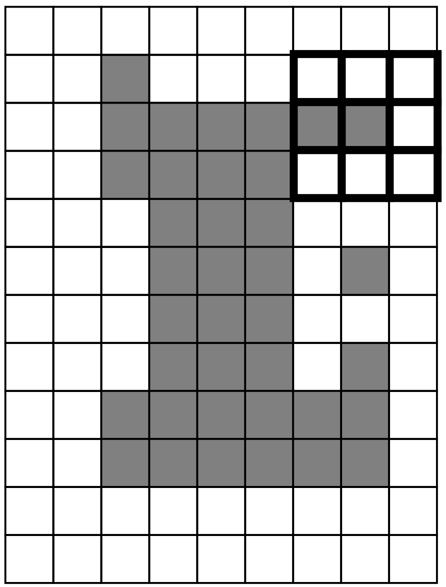

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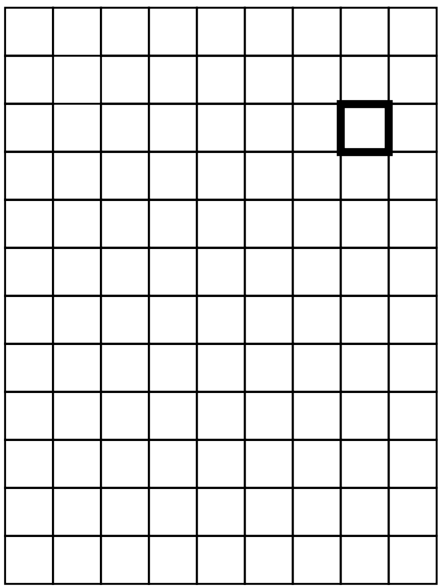

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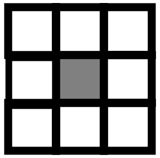

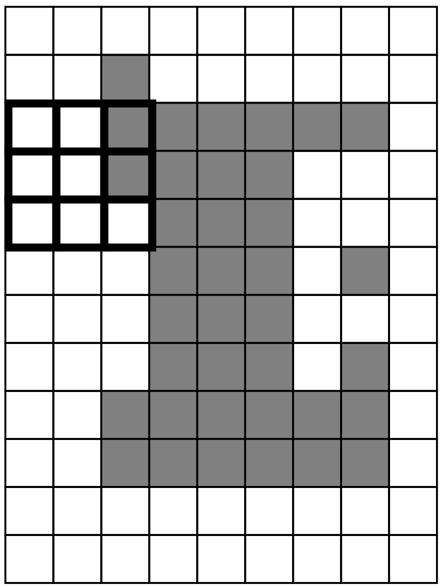

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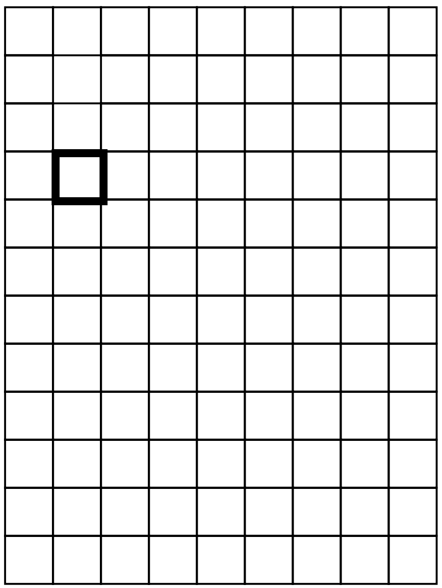

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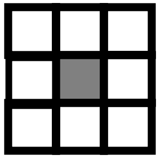

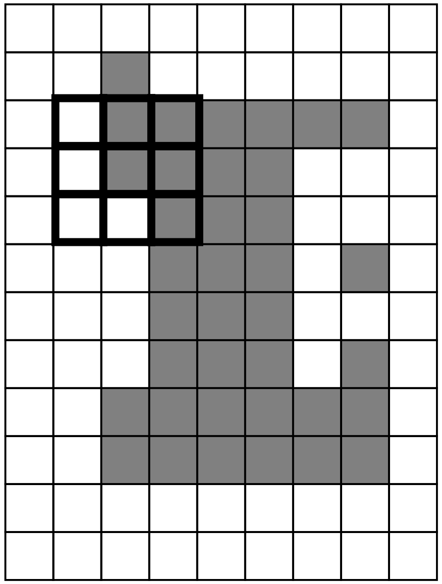

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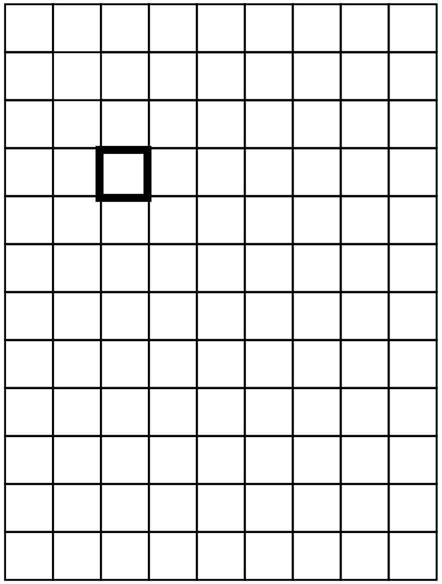

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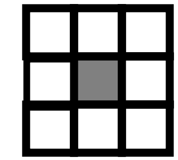
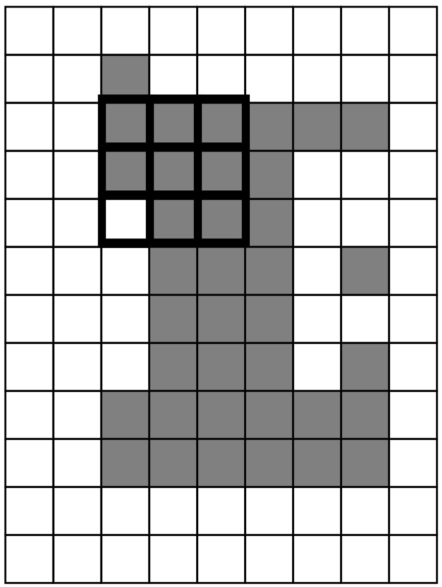
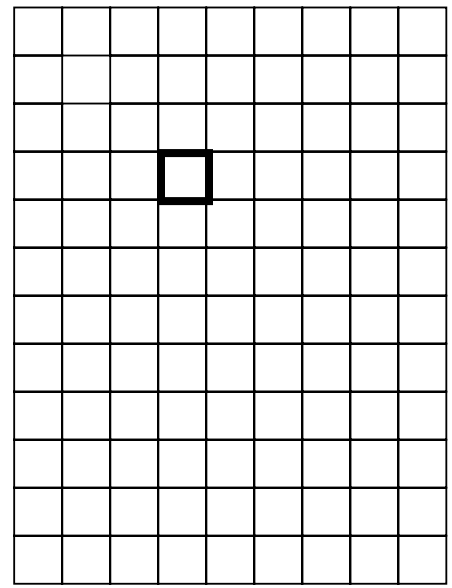




imagine initiala



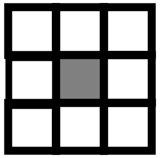
imagine prelucrata

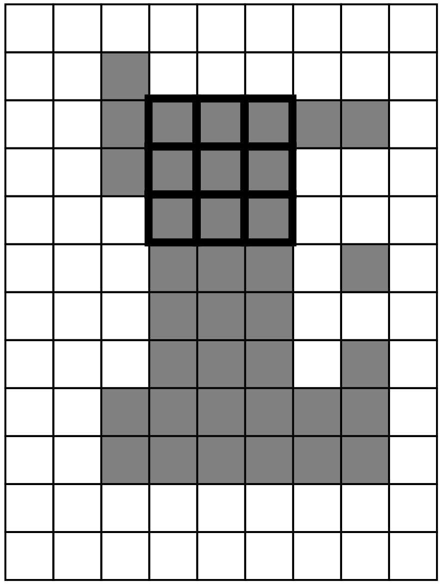
Erodare

$A \,\Theta\, B$

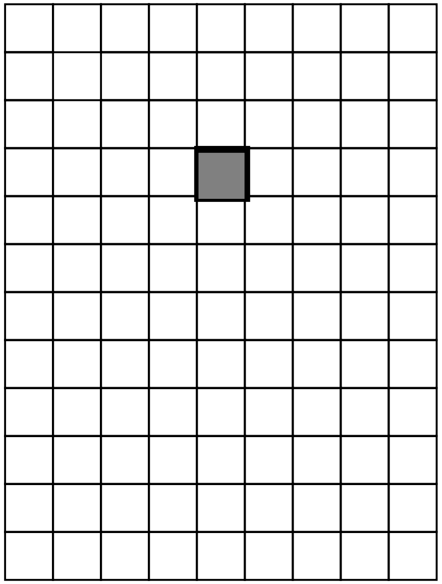
$$A \,\Theta\, B = \{\mathbf{x} \mid B_{\mathbf{x}} \subset A\}$$

$$B = V_8$$





imagine initiala



imagine prelucrata

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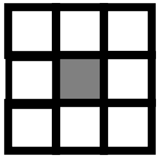
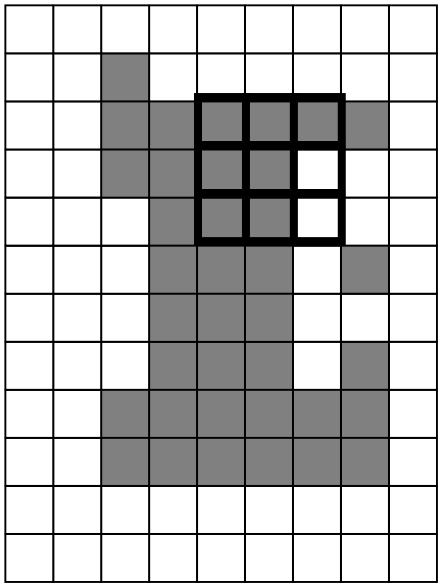
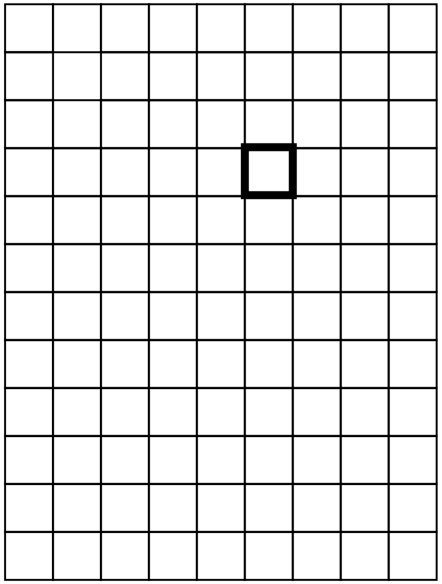




imagine initiala



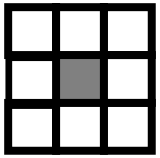
imagine prelucrata

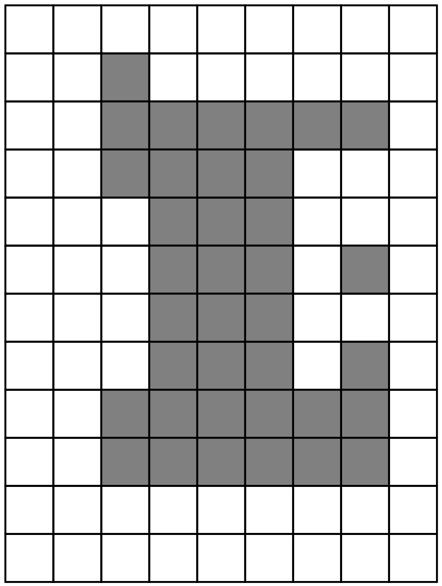
Erodare

$A \ominus B$

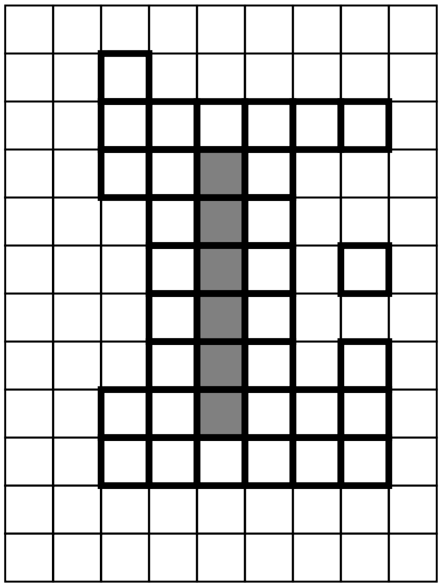
$$A \ominus B = \{\mathbf{x} \mid B_{\mathbf{x}} \subset A\}$$

$$B = V_8$$





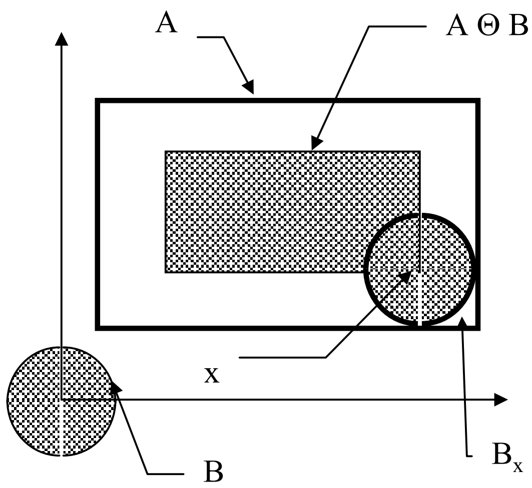
imagine initiala

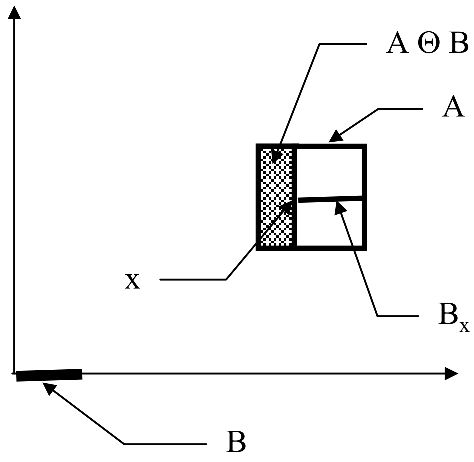


imagine prelucrata

Erodare

$A \ominus B$





Efect principal : micsorare obiecte.

Erodare

$A \Theta B$

Forma echivalenta :

$$A \Theta B = \{\mathbf{x} \mid B_{\mathbf{x}} \subset A\} = \{\mathbf{x} \mid \forall\, \mathbf{b} \in B, \exists\, \mathbf{a} \in A \text{ astfel incat } \mathbf{b} + \mathbf{x} = \mathbf{a}\} =$$

$$= \{\mathbf{x} \mid \forall\, \mathbf{b} \in B, \exists\, \mathbf{a} \in A \text{ astfel incat } \mathbf{x} = \mathbf{a} - \mathbf{b}\} = \bigcap_{\mathbf{b} \in B} A_{-\mathbf{b}} = \bigcap_{\mathbf{b} \in B^S} A_{\mathbf{b}}$$

$B^S = -B$ elementul structurant simetric)

Dilatare

$A \oplus B$

$$A \oplus B = \{\mathbf{x} \mid B_{\mathbf{x}} \cap A \neq \varnothing\}$$

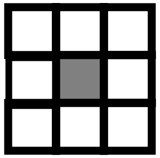
Dilatarea morfologica a multimii *A* prin elementul structurant *B* se defineste ca multimea punctelor (elementelor) cu care (in care) se poate translata elementul structurant astfel incat acesta sa aiba puncte comune cu multimea de prelucrat *A* .

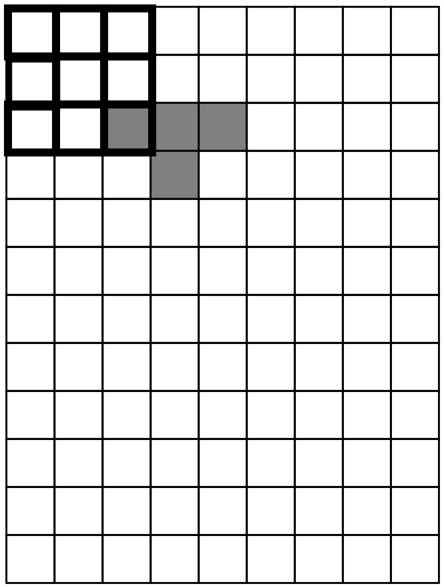
Dilatare

$$A \oplus B = \{\mathbf{x} \mid B_{\mathbf{x}} \cap A \neq \varnothing\}$$

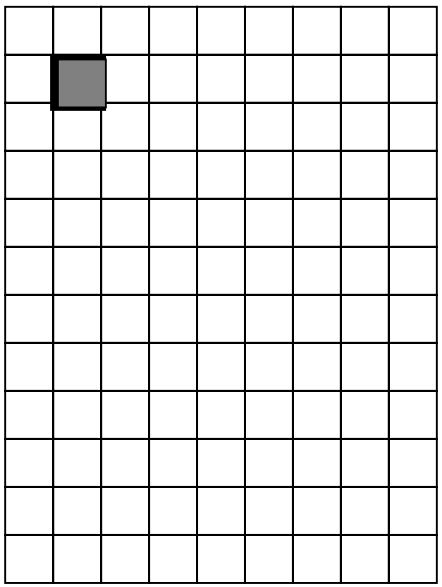
$A \oplus B$

$$B = V_8$$





imagine initiala



imagine prelucrata

Dilatare

$$A \oplus B = \{\mathbf{x} \mid B_{\mathbf{x}} \cap A \neq \emptyset\}$$

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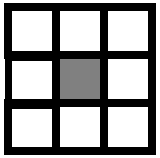
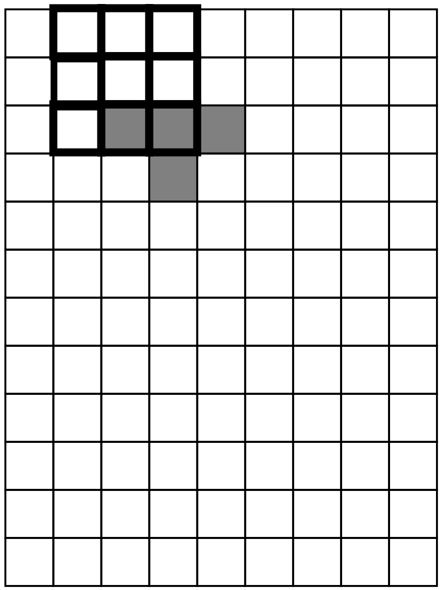
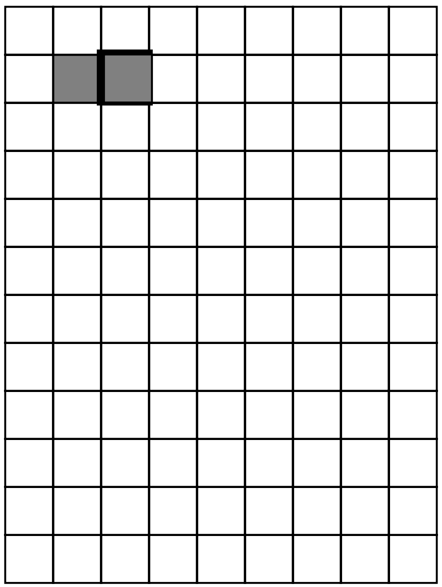




imagine initiala



imagine prelucrata

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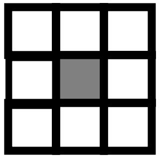

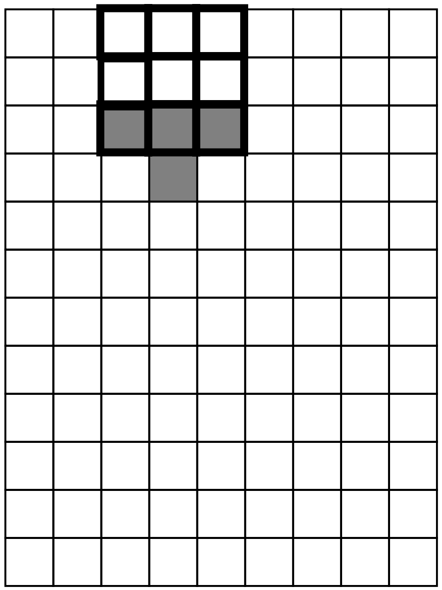

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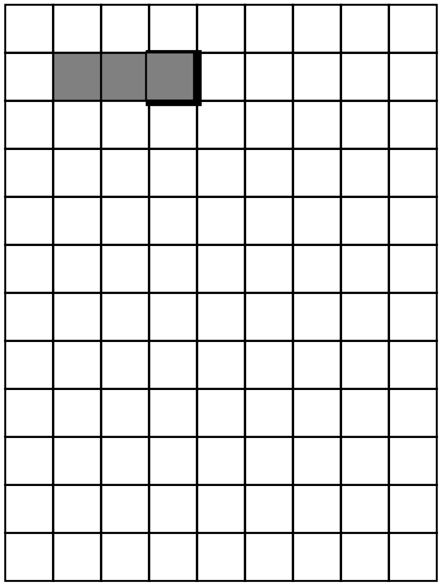

imagine prelucrata

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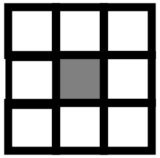
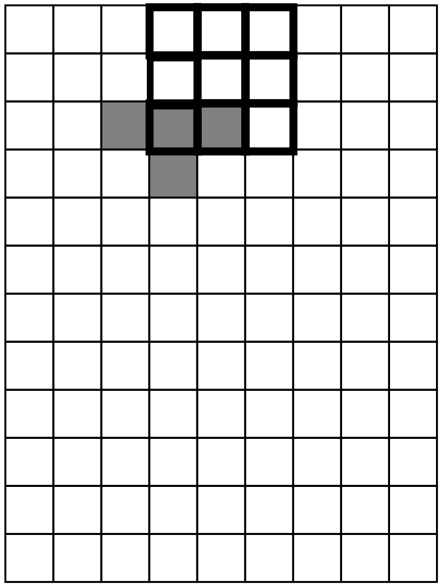
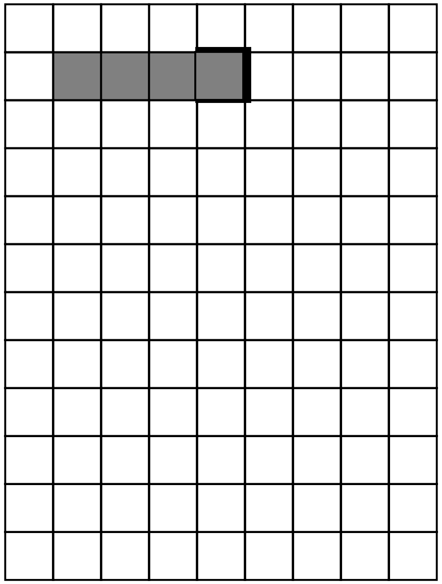




imagine initiala



imagine prelucrata

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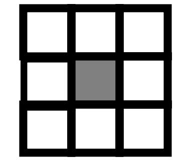

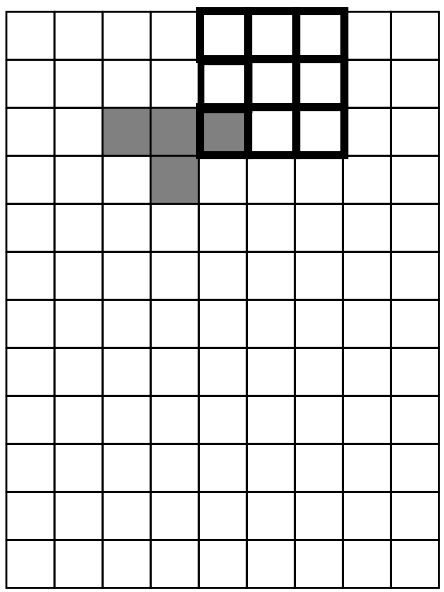

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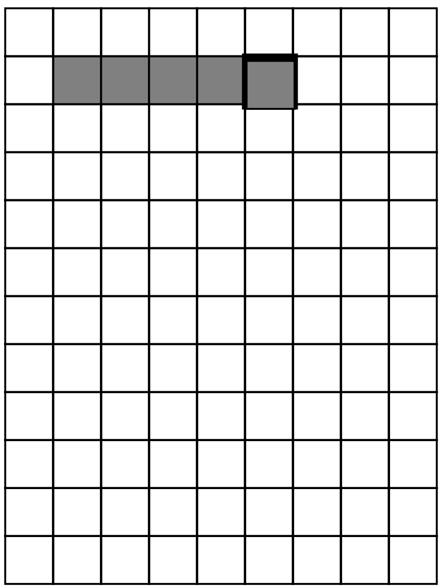

imagine prelucrata

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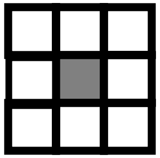

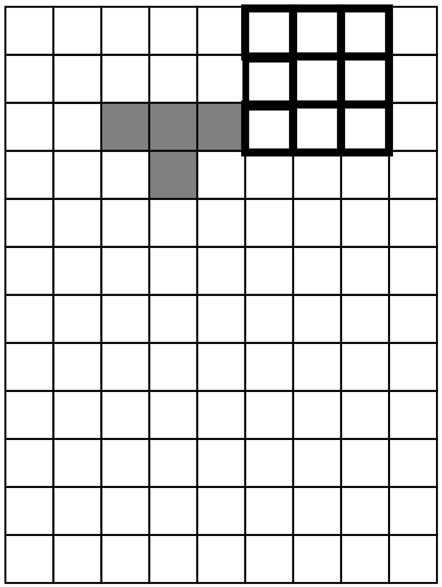

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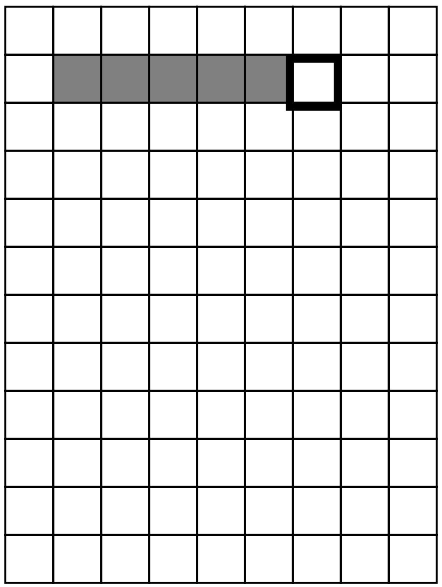

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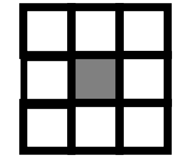

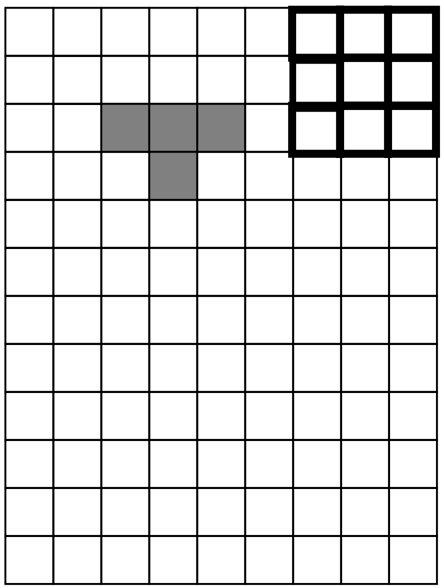

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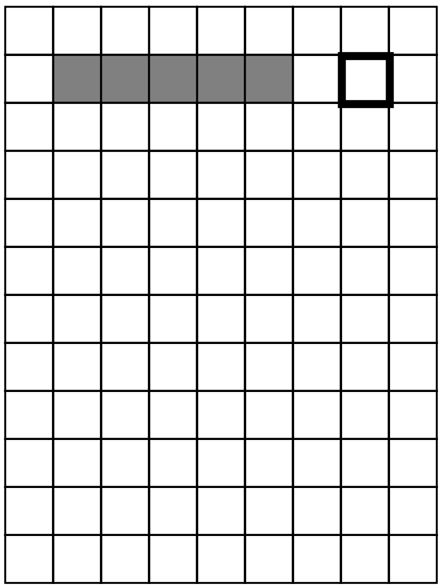

imagine prelucrata

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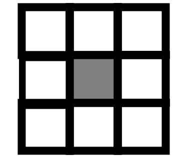

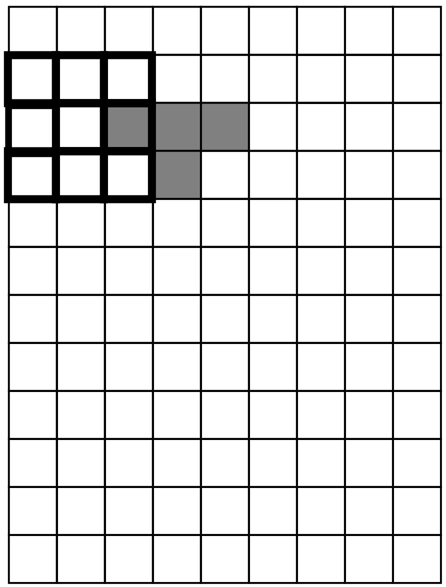

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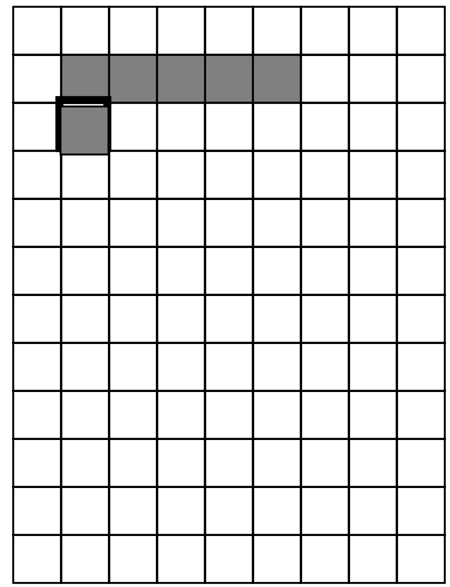

imagine prelucrata

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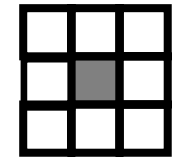
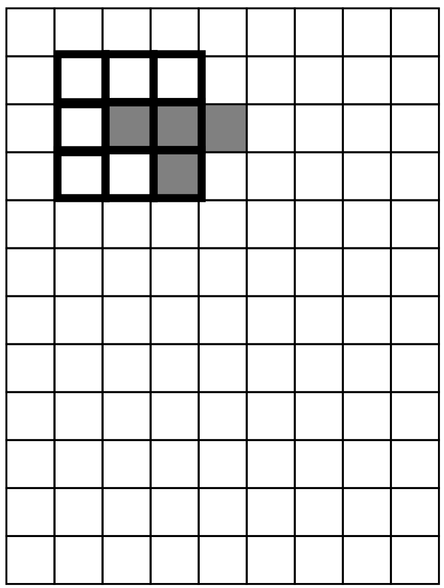
$B = V_8$ 



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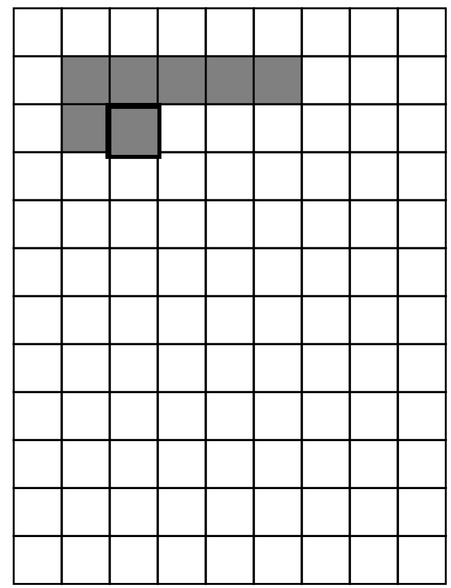

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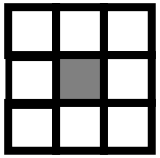

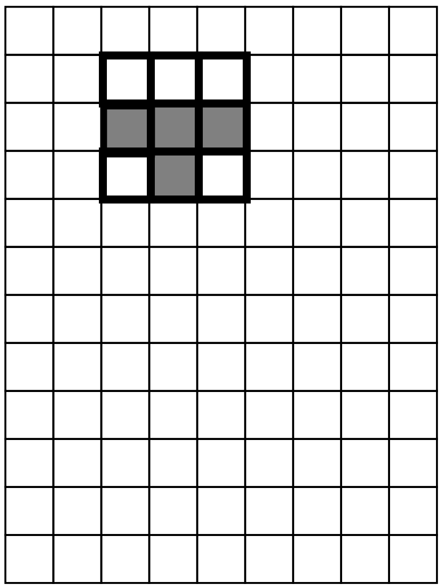

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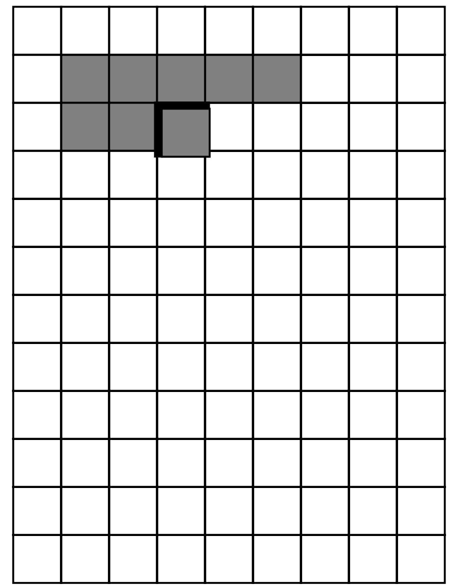

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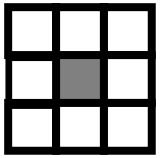

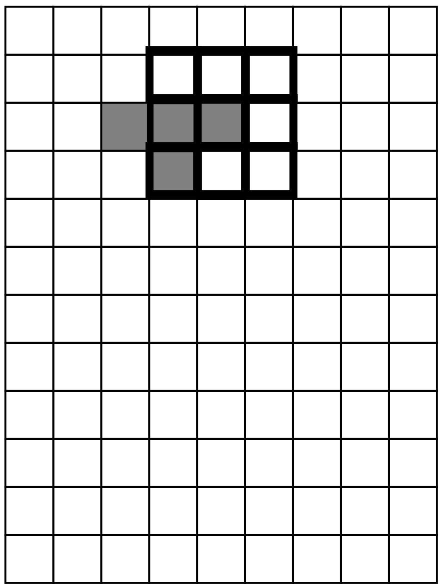

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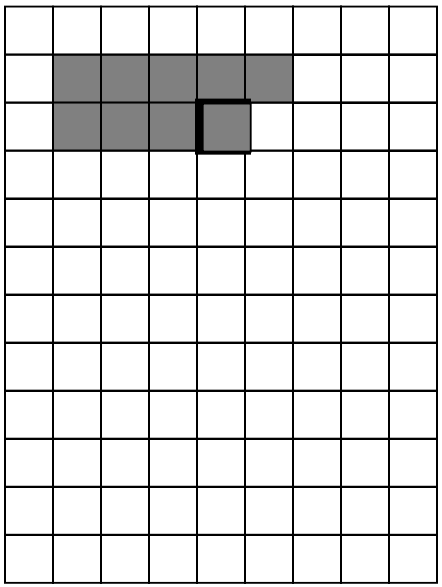

imagine prelucrata

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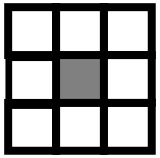

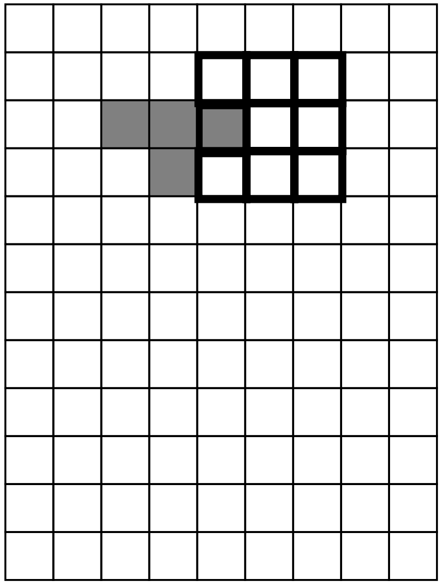

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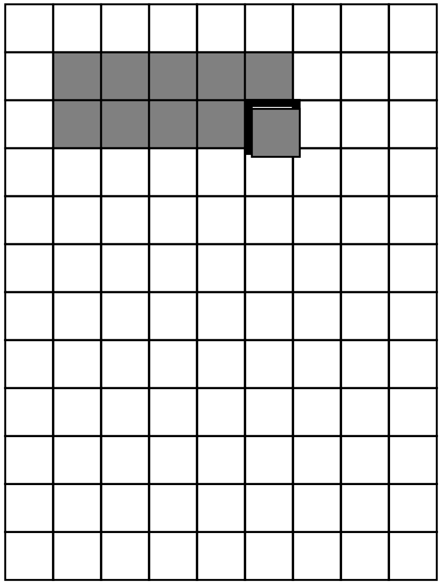

imagine prelucrata

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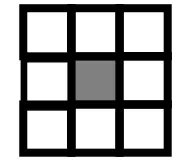

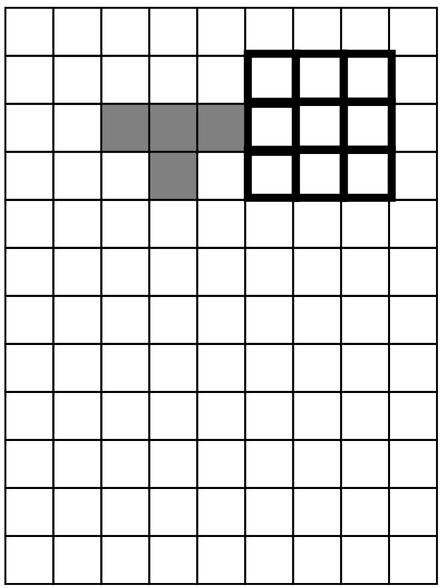

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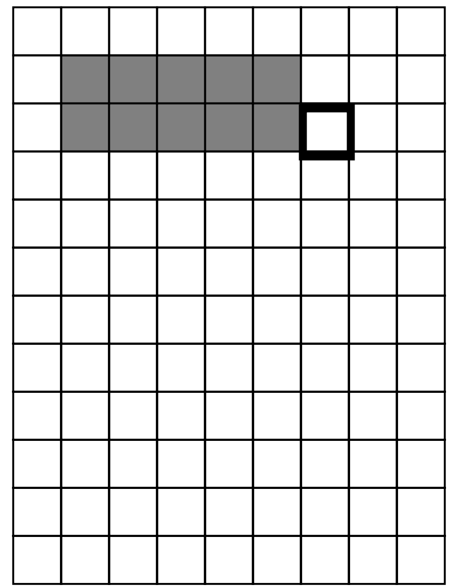

imagine prelucrata

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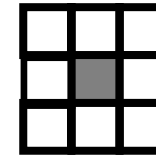
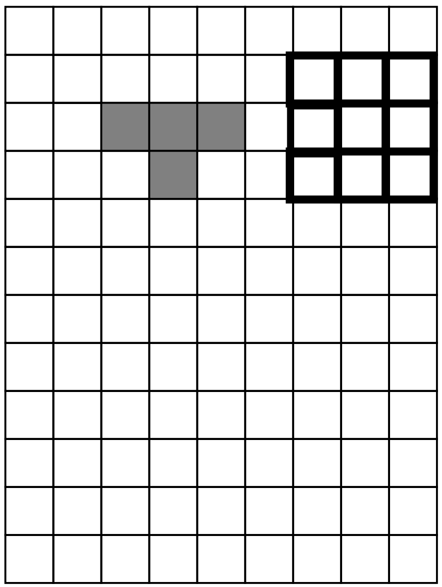
$B = V_8$ 



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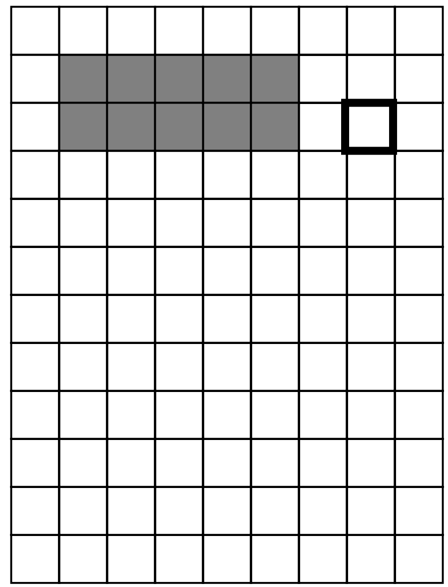

imagine prelucrata

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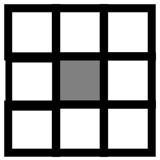

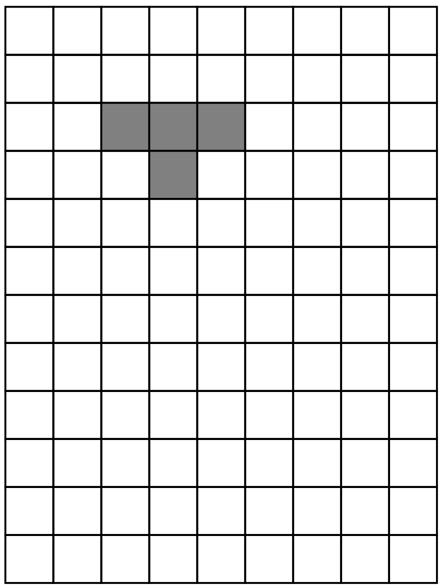

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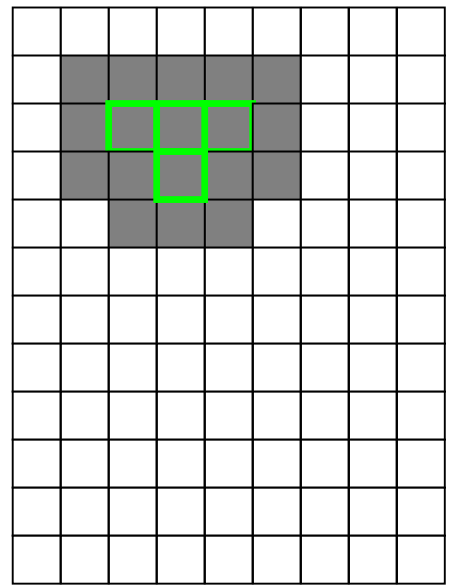
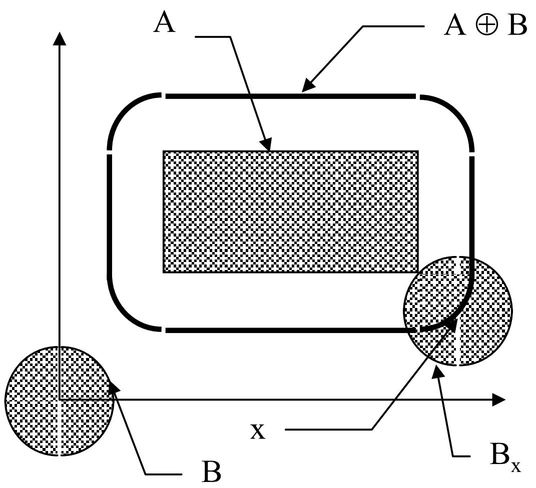
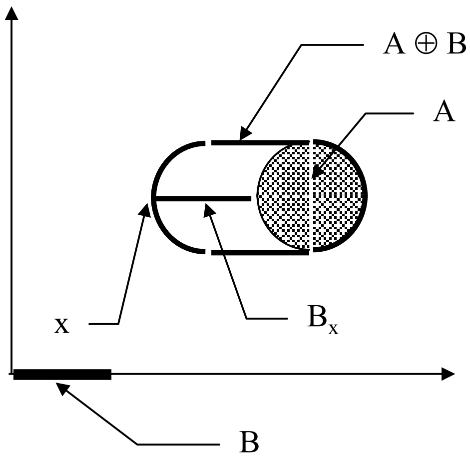


imagine prelucrata

Dilatare

$A \oplus B$





Efect principal : marire obiecte.

Dilatare

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Forma echivalenta :

$$A \oplus B = \{\mathbf{x} \mid B_{\mathbf{x}} \cap A \neq \emptyset\} = \{\mathbf{x} \mid \exists\, \mathbf{b} \in B, \exists\, \mathbf{a} \in A \text{ astfel incat } \mathbf{b} + \mathbf{x} = \mathbf{a}\} =$$

$$= \{\mathbf{x} \mid \exists\, \mathbf{b} \in B, \exists\, \mathbf{a} \in A \text{ astfel incat } \mathbf{x} = \mathbf{a} - \mathbf{b}\} = \bigcup_{\mathbf{b} \in B} A_{-\mathbf{b}} = \bigcup_{\mathbf{b} \in B^S} A_{\mathbf{b}}$$

$B^S = -B$ elementul structurant simetric)

Proprietatile de baza ale erodarii si dilatarii

1. Erodarea si dilatarea **nu sunt inversabile** si **nu sunt inverse una alteia**.

2. Erodarea si dilatarea sunt **duale** in raport cu complementarea multimilor.

$$(A \oplus B)^C = A^C \ominus B$$

$$(A \ominus B)^C = A^C \oplus B$$

Efectele unei transformari asupra obiectelor sunt efectele dualei sale asupra fundalului (multimii duale obiectelor).

Dualitate: demonstratie

$$A \oplus B = \{\mathbf{x} \mid B_{\mathbf{x}} \cap A \neq \varnothing\}$$

$$(A^C \oplus B)^C = \{\mathbf{x} \mid B_{\mathbf{x}} \cap A^C \neq \varnothing\}^C = \{\mathbf{x} \mid B_{\mathbf{x}} \cap A^C = \varnothing\} =$$

$$= \{\mathbf{x} \mid B_{\mathbf{x}} \subset A\} = A \ominus B$$

$$A \ominus B = \{\mathbf{x} \mid B_{\mathbf{x}} \subset A\}$$

$$(A^C \ominus B)^C = \{\mathbf{x} \mid B_{\mathbf{x}} \subset A^C\}^C = \{\mathbf{x} \mid B_{\mathbf{x}} \not\subset A^C\} =$$

$$= \{\mathbf{x} \mid B_{\mathbf{x}} \cap A \neq \varnothing\} = A \oplus B$$

3. Proprietatea de invarianta la translatie

$$A_t \oplus B = (A \oplus B)_t$$

$$A_t \Theta B = (A \Theta B)_t$$

$$A \oplus B_t = (A \oplus B)_{-t}$$

$$A \Theta B_t = (A \Theta B)_{-t}$$

4. Proprietatea de invarianta la scalare

$$1/\lambda\ (\lambda A \oplus B) = A \oplus 1/\lambda B$$

$$1/\lambda\ (\lambda A \Theta B) = A \Theta 1/\lambda B$$

5. Proprietati de monotonie

Transformari crescatoare fata de multimea de prelucrat

$A_1 \subset A_2$ $\quad$ $A_1 \oplus B \subset A_2 \oplus B$

$A_1 \Theta B \subset A_2 \Theta B$

Dilatarea este crescatoare fata de elementul structurant folosit.

$B_1 \subset B_2$ $\quad$ $A \oplus B_1 \subset A \oplus B_2$

Erodarea este descrescatoare fata de elementul structurant folosit.

$B_1 \subset B_2$ $\quad$ $A \Theta B_2 \subset A \Theta B_1$

6. Proprietati de extensivitate

In general dilatarea este extensiva $A \subseteq A \oplus B$

In general erodarea este anti-extensiva. $A \Theta B \subseteq A$

Conditia suficienta pentru ca erodarea sa fie anti-extensiva si dilatarea sa fie extensiva este ca elementul structurant sa isi contina originea (nu este insa si o conditie necesara).

7. “Asociativitatea”

$$A \oplus (B \oplus C) = (A \oplus B) \oplus C^{S}$$

$$(A \ominus B) \ominus C = A \ominus (B \oplus C)$$

“descompunerea” elementului structurant prin dilatare

8. Distributivitatea fata de operatiile pe multimi

$$(A \cup B) \oplus C = (A \oplus C) \cup (B \oplus C)$$

$$A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$$

$$A\oplus (B\cap C) \subset (A\oplus B) \cap(A\oplus C)$$

$$(B\cap C)\oplus A \subset (B\oplus A) \cap(C\oplus A)$$

$$A \Theta (B \cup C) = (A \Theta B) \cap (A \Theta C)$$

$$(A \cap B) \Theta C = (A \Theta C) \cap (B \Theta C)$$

$$A\Theta (B\cap C) \supset (A\Theta B) \cup(A\Theta C)$$

$$(B\cap C) \Theta A \supset (B\Theta A) \cap(C\Theta A)$$

Interesul practic pare relativ limitat ... schimbarile induse formelor sunt prea importante.

Iterarea operatiilor morfologice de baza ?

Deschiderea si inchiderea

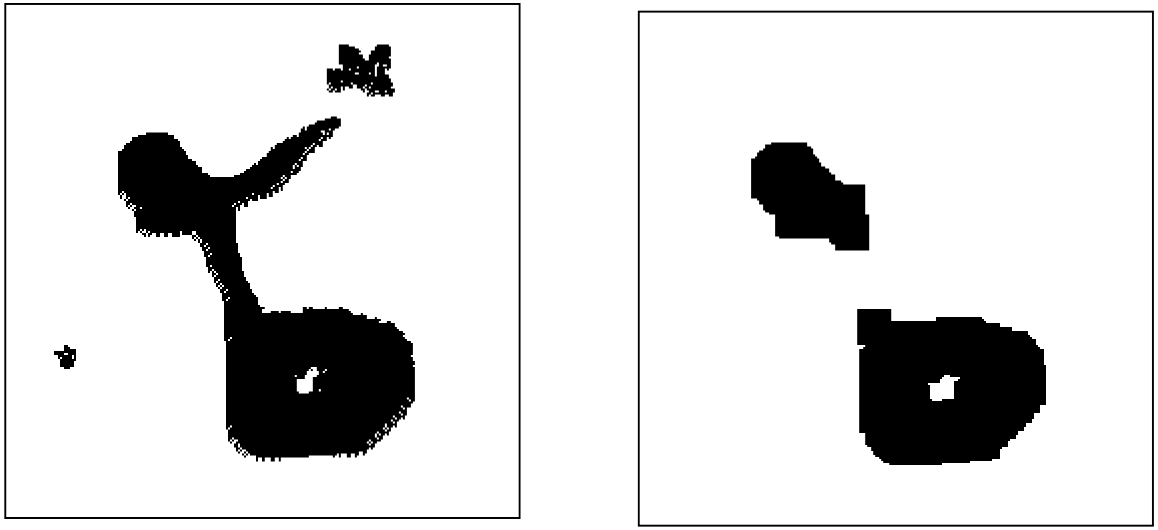
Deschiderea morfologica a multimii A prin elementul structurant B se defineste ca erodarea multimii cu elementul structurant respectiv, urmata de dilatarea cu elementul structurant simetrizat.

$$A \circ B = (A \ominus B) \oplus B^S$$

Inchiderea morfologica a multimii A prin elementul structurant B se defineste ca dilatarea multimii cu elementul structurant respectiv, urmata de erodarea cu elementul structurant simetrizat.

$$A \bullet B = (A \oplus B) \ominus B^S$$

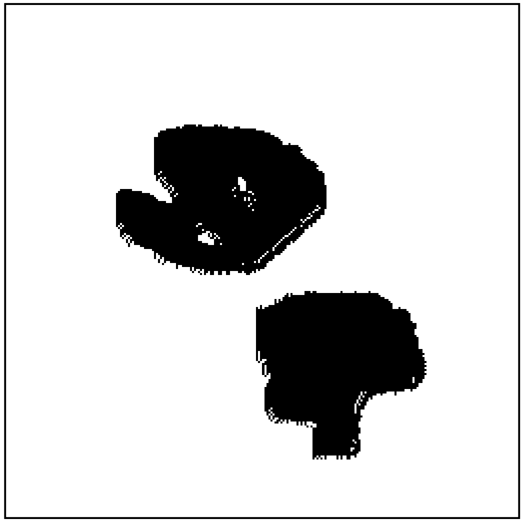
Deschiderea

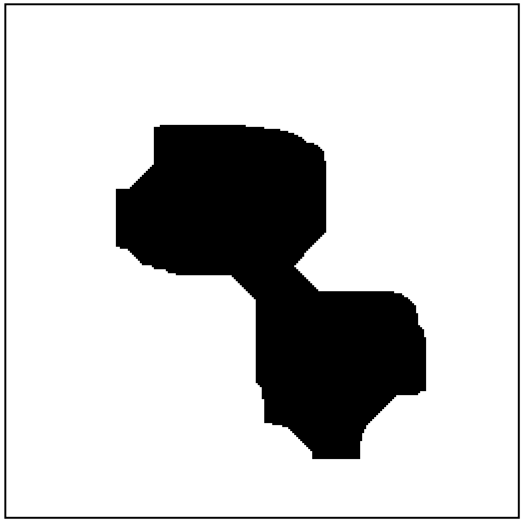


Prin deschidere cu un element structurant disc centrat in origine, componentele conexe ale multimii A mai mici decat elementul structurant sunt indepartate; convexitatile foarte accentuate ale contururilor sunt tesite si "istmurile" sunt indepartate

efect de netezire a formei

Inchiderea





Prin inchidere cu un element structurant disc centrat in origine gaurile incluse in obiecte, mai mici decat elementul structurant folosit sunt umplute, se umplu concavitatile puternice ale contururilor si obiectele foarte apropiate sunt fuzionate

efect de netezire a formei

Proprietati esentiale ale inchiderii si deschiderii

1. Sunt transformari duale una alteia

$$(A \bullet B)^C = A^C \circ B$$

$$(A \circ B)^C = A^C \bullet B$$

2. Deschiderea este anti-extensiva; inchiderea e extensiva

$$A \circ B \subset A \subset A \bullet B$$

3. Sunt transformari idempotente

$$(A \circ B) \circ B = A \circ B$$

$$(A \bullet B) \bullet B = A \bullet B$$

Filtre alternate secvential

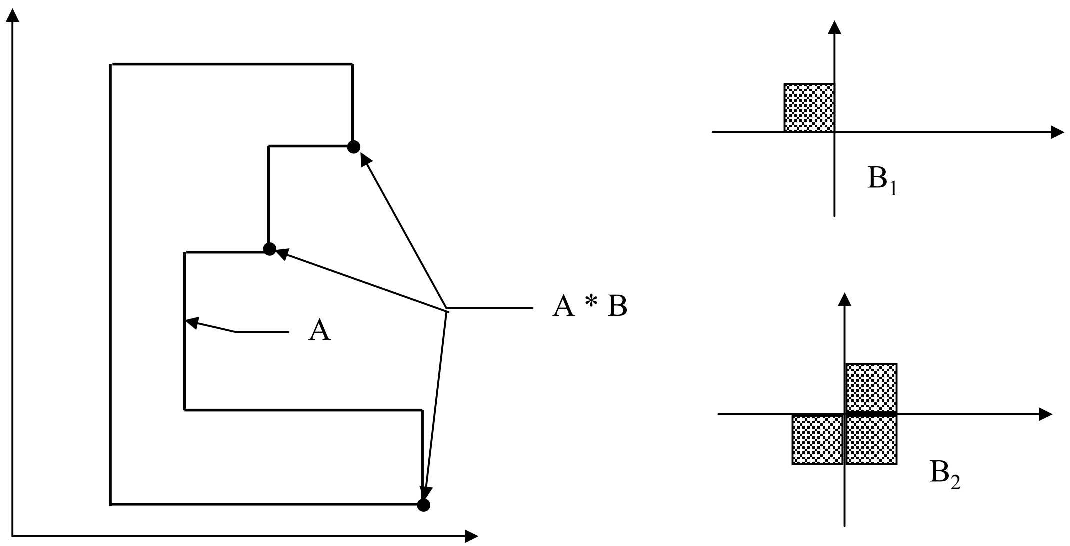
$$FAS(A) = (((((A \circ B) \bullet B) \circ 2B) \bullet 2B) \circ 3B \ldots.$$

$$FAS(A) = (((((A \bullet B) \circ B) \bullet 2B) \circ 2B) \bullet 3B \ldots.$$

Transformarea *Hit or Miss*

$$A * B = (A \ominus B_1) - (A \oplus B_2),$$

cu $B = B_1 \cup B_2$ si $B_1 \cap B_2 = \varnothing$



Extragerea contururilor

Contur exterior $\Delta A = A \oplus B - A$

Contur interior $\delta A = A - A \Theta B$

Gradient morfologic $gradA = A \oplus B - A \Theta B$

Dar pe nivele de gri ?

Erodare = minim

Dilatare = maxim

$$f \ominus g = \min_{y \in Supp(g)} \{ f(x-y) - g(y) \}$$

$$f \oplus g = \max_{y \in Supp(g)} \{ f(x-y) + g(y) \}$$

Pe nivele de gri

Erodare = minim

Dilatare = maxim

$$\text{gradient morfologic} = \max - \min = L - \mathit{filt}$$